Low-energy asymptotics in perturbed periodically twisted quantum waveguides

Daniel Parra

University of Tokyo

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Outline

1. Periodically twisted quantum waveguides
2. Spectral properties of the unperturbed operator H_0.
3. Twisted and bent waveguides
4. Resonances
5. The $\beta = 0$ case
6. Sketch of proof

Periodically twisted quantum waveguides

Let $\omega \subset\in \mathbb{R}^2$ be a bounded domain with a C^2 boundary and let $\beta \in \mathbb{R}$. Define $\mathcal{L}_0 : \mathbb{R} \times \omega \to \mathbb{R}^3$ by

$$\mathcal{L}_0(s,t) = (s, t_2 \cos(\beta s)\mathbf{e_2} - t_3 \sin(\beta s)\mathbf{e_3}, t_2 \sin(\beta s)\mathbf{e_2} + t_3 \cos(\beta s)\mathbf{e_3})$$

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We assume that ω is not rotationally symmetric.

The Dirichlet Laplacian in Ω_0

We consider the Dirichlet Laplacian in Ω_0. Since $\mathcal{L}_0$ is a diffeomorphism, one can see that it is actually unitary equivalent with the operator acting on $L^2(\mathbb{R} \times \omega)$ given by:

$$-H_0 = \Delta_\omega + (\partial_s - \beta \partial_\varphi)^2$$

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where $\partial_\varphi = t_2 \partial_3 - t_3 \partial_2$. Using the partial Fourier transform with respect to the first (longitudinal) variable, denoted by $\mathcal{F}$, we can see that H_0 is unitary equivalent to an analytically fibered operator:

$$\mathcal{F} H_0 \mathcal{F}^{-1} = \int_{\mathbb{R}}^{\oplus} \mathfrak{h}_0(p) dp,$$

where for any $p \in \mathbb{R}$

$$\mathfrak{h}_0(p) = -\Delta_\omega - (ip - \beta \partial_\varphi)^2$$

is an operator acting in $L^2(\omega)$.

Spectral properties of H_0

The family $\{\mathfrak{h}_0(p)\}$ is analytic of type A in the sense of Kato. Since each operator $\mathfrak{h}_0(p)$ has purely discrete spectrum we have a band structure. Furthermore the first eigenvalue $E_1(p)$ is analytic and simple. We can see that the spectrum of H_0 is purely absolutely continuous and given by

$$\sigma(H_0) = \bigcup_{n=1}^{\infty} E_n(\mathbb{R}) = [\mathcal{E}_1, \infty),$$

Furthermore, it is know that there exist an effective mass m_β such that.

$$E_1(p) = \mathcal{E}_1 + m_\beta p^2 (1 + O(p))$$

with $0 < m_\beta \leq 1$, the equality holding if $\beta = 0$.

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$$\mathbf{e_1}(s) = \dot{\gamma}(s), \quad \kappa(s)\mathbf{e_2}(s) = \dot{\mathbf{e}}_\mathbf{1}(s), \quad \mathbf{e_3}(s) = \mathbf{e_1}(s) \times \mathbf{e_2}(s).$$

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We consider a twisting by an angle $\theta(s)$ with respect to this frame defined a twisted frame $\{\mathbf{e}_\mathbf{1}^\theta, \mathbf{e}_\mathbf{2}^\theta, \mathbf{e}_\mathbf{3}^\theta\}$. We assume that it is asymptotically periodic in the sense that it satisfies $\dot{\theta}(s) = \varepsilon(s) + \beta$ with ϵ decaying at infinity. Then we can define

$$\mathcal{L}(s, t_2, t_3) = \gamma(s) + t_2\mathbf{e}_\mathbf{2}^\theta(s) + t_3\mathbf{e}_\mathbf{3}^\theta(s)$$

and set accordingly $\Omega := \mathcal{L}(\mathbb{R} \times \omega)$.

The Dirichlet Laplacian in Ω

If we assume that $\mathcal{L}$ is injective and $||\kappa||_{L^\infty} \sup_{t\in\omega} |t| < 1$, $\mathcal{L}$ is a diffeomorphism between the straight tube $\mathbb{R} \times \omega$ and Ω.

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If we assume that $\mathcal{L}$ is injective and $||\kappa||_{L^\infty} \sup_{t\in\omega} |t| < 1$, $\mathcal{L}$ is a diffeomorphism between the straight tube $\mathbb{R} \times \omega$ and Ω. Implementing this equivalence, we can see that the Dirichlet Laplacian in Ω is equivalent with the operator in $L^2(\mathbb{R} \times \omega)$ defined by

$$-H = \Delta_\omega + \frac{\kappa^2}{4h^2} + \left(h^{-1/2}(\partial_s + (\tau - \dot{\theta})\partial_\varphi)h^{-1/2}\right)^2$$

where

$$h(s,t) = 1 - \kappa(s)(t_2 \cos(\theta(s) + t_3 \sin(\theta(s))))$$

is the square root of the determinant of the metric tensor induced by $\mathcal{L}$.

The coupling constant δ

We want to make our deformed tube depending on a coupling constant. For $\delta \in [0,1]$ consider

$$\dot{\theta}_\delta = \beta + \delta\varepsilon, \quad \kappa_\delta = \delta\kappa \quad \text{and} \quad \tau_\delta = \delta\tau.$$

We define in $L^2(\mathbb{R} \times \omega)$ the operator given by particularizing the expression for H to $\theta_\delta, \kappa_\delta$ and τ_δ

$$H_\delta = -\Delta_\omega - \frac{\delta^2\kappa^2}{4h_\delta^2} - \left(h_\delta^{-1/2}(\partial_s + (\delta(\tau - \varepsilon) - \beta)\,\partial_\varphi)h_\delta^{-1/2}\right)^2,$$

where $h_\delta(s,t) = 1 - \delta\kappa(s)\big(t_2 \cos\theta_\delta(s) + t_3 \sin\theta_\delta(s)\big)$.

Assumptions (1)

We assume that $\varepsilon, \kappa, \tau : \mathbb{R} \to \mathbb{R}$ are functions of class C^2 with exponential decay i.e., for some $\alpha > 2(\mathcal{E}_2 - \mathcal{E}_1)^{\frac{1}{2}} > 0$ they satisfy

$$\kappa(s), \tau(s), \varepsilon(s) = O(e^{-\alpha\langle s\rangle}),$$

where $\langle s\rangle := (1 + s^2)^{1/2}$, and the same assumption for their first and second derivatives. Setting

$$W(\delta) = H_\delta - H_0$$

we can see that that $W(\delta)$ is a second order differential operator with decaying coefficients for every δ. Under this assumptions one can show that $H_\delta^{-1} - H_0^{-1}$ is compact, and hence conclude that

$$\sigma_{\text{ess}}(H_\delta) = [\mathcal{E}_1, \infty).$$

Assumptions (2)

Let η be an exponential weight of the form

$$\eta(s) = e^{-N\langle s \rangle} \quad \text{with} \quad (\mathcal{E}_2 - \mathcal{E}_1)^{\frac{1}{2}} < N < \frac{\alpha}{2}.$$

The function

$$k \mapsto \left((H_\delta - \mathcal{E}_1 - k^2)^{-1} : \eta L^2(\mathbb{R} \times \omega) \to \eta^{-1} D(H) \subset \eta^{-1} L^2(\mathbb{R} \times \omega) \right),$$

initially defined for k in $\mathbb{C}^{++} := \{k \in \mathbb{C};\ \operatorname{Im} k > 0;\ \operatorname{Re} k > 0\}$, is clearly analytic

Theorem ($\beta \neq 0$)

Let $\varepsilon, \kappa, \tau : \mathbb{R} \to \mathbb{R}$ be non-zero C^2-functions as above and fix a sufficiently small neighborhood of zero $\mathcal{D}$ in $\mathbb{C}$. Then, there exists $\delta_0 > 0$ such that for $\delta \leq \delta_0$, the analytic operator-valued function

$$k \mapsto \left((H_\delta - \mathcal{E}_1 - k^2)^{-1} : \eta L^2(\mathbb{R} \times \omega) \mapsto \eta^{-1} L^2(\mathbb{R} \times \omega)\right)$$

admits a meromorphic extension on $\mathcal{D}$. This function has a unique pole $k(\delta)$ in $\mathcal{D}$, which has multiplicity one. Moreover, we have

$$k(\delta) = i\mu_1 \delta + O(\delta^2),$$

where

$$\mu_1 = \frac{\beta}{\sqrt{m_\beta}} \int_{\mathbb{R}\times\omega} (\tau - \varepsilon)|\partial_\varphi \psi_1|^2 - \frac{\beta^2}{\sqrt{m_\beta}} \int_{\mathbb{R}\times\omega} \kappa\left(|\partial_\varphi \psi_1|^2 + \tfrac{1}{4}|\psi_1|^2\right)\vartheta,$$

with $\vartheta(s,t) := (t_2 \cos(\beta s) + t_3 \sin(\beta s))$. Furthermore, the pole $k(\delta)$ is a purely imaginary number.

Comments

- In terms of the resolvent $(H_\delta - z)^{-1}$ we can say that it admits a meromorphic extension in a neighborhood of $\mathcal{E}_1$ in the 2-sheeted Riemann surface of $\sqrt{z - \mathcal{E}_1}$.

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- If $\mu_1 > 0$, the pole $k(\delta)$ lies in the positive imaginary axis and correspond to a real resonance under the essential spectrum (in particular it gives raise to an eigenvalue).
- If $\mu_1 < 0$, the pole $k(\delta)$ lies in the negative imaginary axis and correspond to a resonance that lies in the second sheet of the 2-sheeted Riemann surface. In particular this kind of resonances are far from the real axis and are sometimes refer as anti-bound states.

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- If $\mu_1 = 0$ we need to go the next order in order to understand the nature of the resonance.

Theorem ($\beta = 0$)

Let us set $\beta = 0$. Let $\varepsilon, \kappa, \tau : \mathbb{R} \to \mathbb{R}$ be non-zero C^2-functions satisfying the above stated decay. Fix a sufficiently small neighborhood of zero $\mathcal{D}$ in $\mathbb{C}$. Then, there exists $\delta_0 > 0$ such that for $\delta \leq \delta_0$, the analytic operator-valued function

$$k \mapsto \Big((H_\delta - \mathcal{E}_1 - k^2)^{-1} : \eta L^2(\mathbb{R} \times \omega) \to \eta^{-1} L^2(\mathbb{R} \times \omega) \Big)$$

admits a meromorphic extension on $\mathcal{D}$. This function has a unique pole $k(\delta)$ in $\mathcal{D}$, which has multiplicity one. Moreover

$$k(\delta) = i\mu_2 \delta^2 + O(\delta^3),$$

where

$$\begin{aligned}
\mu_2 =& \frac{1}{8} \sum_{q>1} (\mathcal{E}_q - \mathcal{E}_1)^2 \langle \psi_q | t_2 \psi_1 \rangle^2 \langle \kappa | (D_s^2 + \mathcal{E}_q - \mathcal{E}_1)^{-1} \kappa \rangle \\
& - \frac{1}{2} \sum_{q>1} (\mathcal{E}_q - \mathcal{E}_1) \langle \psi_q | \partial_\varphi \psi_1 \rangle^2 \langle (\tau - \varepsilon) | (D_s^2 + \mathcal{E}_q - \mathcal{E}_1)^{-1} (\tau - \varepsilon) \rangle \\
& - \frac{1}{2} \sum_{q>1} (\mathcal{E}_q - \mathcal{E}_1) \langle \pi_q t_2 \psi_1 | \partial_\varphi \psi_n \rangle \langle \tau - \varepsilon | (D_s^2 + \mathcal{E}_q - \mathcal{E}_1)^{-1} \dot{\kappa} \rangle.
\end{aligned}$$

Scketch of proof (1)

- Localizing in energy and using the properties of the first band $\mathcal{E}_1$ one can first show that

$$\eta R_0(k)\eta := \eta(H_0 - \mathcal{E}_1 - k^2)^{-1}\eta = \frac{A_{-1}}{k} + F(k)$$

where $F(k)$ is analytic in a some small neighborhood $\mathcal{D}$ and A_{-1} is a rank one operator.

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  where $F(k)$ is analytic in a some small neighborhood $\mathcal{D}$ and A_{-1} is a rank one operator.

- Since in $\mathbb{C}^{++}$ we have

$$\eta(H_\delta - \mathcal{E}_1 - k^2)^{-1}\eta = \eta R_0(k)\eta \left(\mathrm{Id} + \eta^{-1}W_\delta\eta^{-1}\eta R_0(k)\eta\right)^{-1}$$

  we need to study $\eta^{-1}W_\delta\eta^{-1}\eta R_0(k)\eta$.

Scketch of proof (2)

Lemma

There exists a neighborhood $\mathcal{D}$ of zero and $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ and $k \in \mathcal{D} \setminus \{0\}$

$$\eta^{-1} W_\delta \eta^{-1} \eta R_0(k) \eta = \frac{1}{k} K_0(\delta) + T(\delta, k),$$

where $K_0(\delta)$ is the rank one operator

$$K_0(\delta) = \frac{i}{2\sqrt{m_\beta}} |\eta^{-1} W_\delta \psi_1\rangle\langle \eta \otimes \psi_1|,$$

and $\mathcal{D} \ni k \mapsto (T(\delta, k)\colon L^2(\mathbb{R} \times \omega) \to L^2(\mathbb{R} \times \omega))$ is an analytic operator-valued function. Moreover,

$$\sup_{k \in \mathcal{D}} ||T(\delta, k)|| = O(\delta).$$

Scketch of proof (3)

To end the proof one can see that

$$\Big(\mathrm{Id} + \eta^{-1} W_\delta R_0(k)\eta\Big) = \Big(\mathrm{Id} + T(\delta, k)\Big)\Big(\mathrm{Id} + \frac{1}{k}(\mathrm{Id} + T(\delta, k))^{-1} K_0(\delta)\Big)$$

implies that by setting

$$\nu_\delta(k) := \langle \eta\psi_1 | \frac{1}{k}(\mathrm{Id} + T(\delta, k))^{-1} \frac{i}{2\sqrt{m_\beta}} \eta^{-1} W_\delta \psi_1 \rangle$$

there exist projections $\Pi_\delta^\perp$ and Π_δ we have

$$\eta R(k)\eta = \frac{1}{k + \nu_\delta(k)}\Big(A_{-1} + kF(k)\Big)\Pi_\delta(\mathrm{Id} + T(\delta, k))^{-1} + F(k)\Pi_\delta^\perp(\mathrm{Id} + T(\delta, k))^{-1}.$$