Quantum differentiability on quantum tori

Fedor Sukochev
Joint with E. McDonald and X. Xiong.

University of New South Wales

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(a) An analogy of classical calculus developed for noncommutative settings
(b) A rigorous treatment of "infinitesimal" quantities
(c) A form of calculus which is well-suited to certain "non-smooth" settings.

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- The space ℓ_p for $p \in (0,\infty)$ is the space of p-summable sequences.

Infinitesimals

In historical mathematics, a heuristic infinitesimal is supposed to have the following property:

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Sizes of infinitesimals

The size of an infinitesimal T is described as the rate of decay of its singular value sequence $\{\mu_k(T)\}_{k=0}^{\infty}$.

$$\begin{aligned} \mu_k(T) \text{ is of finite support} &\quad\leftrightarrow\quad T \text{ if of finite rank} \\ \mu_k(T) \in \ell_p &\quad\leftrightarrow\quad T \in \mathcal{L}_p \\ \mu_k(T) = O(k^{-\frac{1}{p}}) &\quad\leftrightarrow\quad T \in \mathcal{L}_{p,\infty} \\ \{k^{\frac{1}{p}-\frac{1}{q}}\mu_k(T)\} \in \ell_q &\quad\leftrightarrow\quad T \in \mathcal{L}_{p,q} \end{aligned}$$

Sometimes $T \in \mathcal{L}_{p,\infty}$ is stated as "T is an infinitesimal of order $\frac{1}{p}$".

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1. a $*$-algebra representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,
2. an operator $F = F^*$, $F^2 = 1$, on $\mathcal{H}$ such that $[F, \pi(a)]$ is a compact operator on $\mathcal{H}$ for any $a \in \mathcal{A}$.

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(this definition has its origins with M. Atiyah's definition of an "abstract elliptic operator" and has been heavily studied in noncommutative geometry)

• **Quantized calculus** of differential forms: a quantised differential $đf$ of $f \in \mathcal{A}$ is defined as:

$$đf := i[F, \pi(f)] = i(F\pi(f) - \pi(f)F) \in K(\mathcal{H});$$

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$$\mathrm{Tr}_\omega(T) := \omega\Big(\Big\{ \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu_k(T) \Big\}_{n=0}^{\infty} \Big).$$

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Tr_ω vanishes on any infinitesimal of order larger than 1, and moreover on $\mathcal{L}_1$.
Connes suggests that $\mathrm{Tr}_\omega(T)$ should be interpreted as "the integral of the order 1 infinitesimal T".

Other traces on $\mathcal{L}_{1,\infty}$.

Dixmier's trace is called a trace due to satisfying the property:

$$\mathrm{Tr}_\omega(BA) = \mathrm{Tr}_\omega(AB), \quad A \in \mathcal{L}_{1,\infty}, B \in B(\mathcal{H}).$$

There are many other linear functionals $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ which satisfy this property.

Of particular interest in this talk are continuous traces. A trace φ is called continuous if:

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Dixmier traces are continuous.

The unit circle

We can get a tractable special case by considering the unit circle:

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- **Fredholm module:** $\mathcal{A} = C(\mathbb{T}), \mathcal{H} = L_2(\mathbb{T})$.
- F is the Hilbert transform: for $g = \sum_{n\in\mathbb{Z}} \hat{g}(n)z^n \in L_2(\mathbb{T})$,

$$Fg = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\hat{g}(n)z^n$$

The unit circle

•

$\bar{d}f$ is related to a Hankel operator:

1. Nehari 1957: $\bar{d}f$ is bounded iff f has bounded mean oscillation ($\sup_I \frac{1}{|I|}\int_I |f - f_I| ds < \infty$)
2. Coifman-Rochberg-Weiss 1976: $\bar{d}f$ is compact iff $f \in VMO$ ($\lim_{|I|\to 0} \frac{1}{|I|}\int_I |f - f_I| ds = 0$)
3. Peller 1980: $\bar{d}f \in \mathcal{L}_p$ iff f is in a certain Besov space ($B^{\frac{1}{p}}_{p,p}(\mathbb{T})$)
4. With real interpolation it is possible to obtain necessary and sufficient conditions for $\bar{d}f \in \mathcal{L}_{p,q}$.

d-Torus $\mathbb{T}^d$ ($d \geq 2$)

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The d-dimensional version of the Hilbert transform is a Riesz transform R_j, $j = 1, \ldots, d$, defined on a Fourier basis element $z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d}$ by:

$$R_j(z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d}) = \frac{n_j}{(n_1^2 + n_2^2 + \cdots + n_d^2)^{1/2}} z_1^{n_1} \cdots z_d^{n_d}.$$

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Let $\{\gamma_j\}_{j=1}^d$ be the Euclidean γ matrices in dimension d. That is, specific $N \times N$ matrices satisfying $\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{j,k} 1$ where $N = 2^{\lfloor d/2 \rfloor}$.

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This is a Fredholm module: by construction we have $F = F^*$ and $F^2 = 1$.

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2. Janson-Wolff 1982: For $p > d$, $đf \in \mathcal{L}_p$ iff $f \in B^{\frac{d}{p}}_{p,p}$; for $p \leq d$, $đf \in \mathcal{L}_p$ iff f is constant.
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- Similar results hold for $\mathbb{R}$ and $\mathbb{R}^d$.

$\bar{d}f \in \mathcal{L}_{d,\infty}$ for f on $\mathbb{R}^d$ (or $\mathbb{T}^d$)

Why do we describe $\bar{d}f \in \mathcal{L}_{d,\infty}$?
This condition is important in noncommutative differential geometry. If $\bar{d}f$ is in $\mathcal{L}_{d,\infty}$ then we can form the "integral"

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- Connes-Sullivan-Teleman 1994: For locally integrable f, $đf \in \mathcal{L}_{d,\infty}$ iff $f \in \dot{W}^1_d$.
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$(\|f\|_{\dot{W}^1_d} = \|\partial_1 f\|_d + \cdots + \|\partial_d f\|_d)$
- Lord-McDonald-S-Zanin 2017 give a different proof. Moreover,

$$\varphi(|\bar{d}f|^d)^{\frac{1}{d}} = C_d \|\big(\sum_{1\leq j\leq d} |\partial_j f|^2\big)^{\frac{1}{2}}\|_d$$

for any continuous normalized trace φ on $\mathcal{L}_{1,\infty}$.

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• **Trace:** Let $\mathcal{P}_\theta$ denote the involutive subalgebra of polynomials in $U_1, U_2, \ldots, U_d$, dense in $C(\mathbb{T}^d_\theta)$. For any polynomial $x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m$ define $\tau(x) = \alpha_{\mathbf{0}}$. Then τ extends to a faithful tracial state on $C(\mathbb{T}^d_\theta)$.

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$L_\infty(\mathbb{T}^d_\theta)$ is the unique hyperfinite type II_1 factor (for "typical" θ).

Calculus on quantum tori

The trace τ is like an integral, and there is a noncommutative measure theory:

• **Noncommutative** L_p**-spaces:** For $1 \leq p < \infty$ and $x \in L_\infty(\mathbb{T}_\theta^d)$ let $\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}$ with $|x| = (x^*x)^{\frac{1}{2}}$. This defines a norm on $L_\infty(\mathbb{T}_\theta^d)$.
The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$.
Note that $L_2(\mathbb{T}_\theta^d)$ is exactly the GNS space of τ.

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Note that $L_2(\mathbb{T}^d_\theta)$ is exactly the GNS space of τ.

• **Partial derivatives on quantum tori:** For $j = 1, \ldots, d$, the jth "partial derivative ∂_j can be defined by:

$$\partial_j(U_1^{n_1} U_2^{n_2} \cdots U_d^{n_d}) = i n_j U_1^{n_1} \cdots U_d^{n_d}.$$

Each ∂_j defines a derivation $\partial_j : \mathcal{P}_\theta \to \mathcal{P}_\theta$.

Sobolev spaces on quantum tori

It is possible to define, for $1 \leq p \leq \infty$, the quantum torus Sobolev space $W^k_p(\mathbb{T}^d_\theta)$ which has a norm:

$$\|x\|_{W^k_p(\mathbb{T}^d_\theta)} = \sum_{|\alpha|\leq k} \|\partial^\alpha x\|_p$$

Similarly there are homogeneous Sobolev spaces:

$$\|x\|_{\dot{W}^k_p(\mathbb{T}^d_\theta)} = \sum_{0<|\alpha|\leq k} \|\partial^\alpha x\|_p.$$

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• **Construction of** F**:** $D_j = -i\partial_j$ are self-adjoint, so is $\mathcal{D} = \sum_j \gamma_j \otimes D_j$. By functional calculus

$$F = \mathrm{sgn}(\mathcal{D}) = \sum_j \gamma_j \otimes \frac{D_j}{\sqrt{D_1^2 + \cdots + D_d^2}}.$$

Then $F = F^*$, $F^2 = 1$.

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Then $F = F^*$, $F^2 = 1$.

• **Quantum derivative:** Let $M_x : y \mapsto xy$ left multiplication representing $C(\mathbb{T}^d_\theta)$ on $L_2(\mathbb{T}^d_\theta)$.

$$\bar{d}x = i[F, M_x].$$

Main results

Theorem (McDonald-S-Xiong. 2018)

For $x \in L_2(\mathbb{T}^d_\theta)$, $\bar{d}x \in \mathcal{L}_{d,\infty}$ iff $x \in \dot{W}^1_d(\mathbb{T}^d_\theta)$.

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Theorem (McDonald-S-Xiong. 2018)

For $x \in L_2(\mathbb{T}_\theta^d)$, $\bar{d}x \in \mathcal{L}_{d,\infty}$ iff $x \in \dot{W}_d^1(\mathbb{T}_\theta^d)$.

Theorem (McDonald-S-Xiong. 2018)

Let $x \in L_2(\mathbb{T}_\theta^d) \cap \dot{W}_d^1(\mathbb{T}_\theta^d)$ be self-adjoint. For any continuous normalized trace φ on $\mathcal{L}_{1,\infty}$ we have

$$\varphi(|\bar{d}x|^d) = c_d \int_{\mathbb{S}^{d-1}} \tau\Big(\big(\sum_{j=1}^{d} |\partial_j x - s_j \sum_{k=1}^{d} s_k \partial_k x|^2\big)^{\frac{d}{2}}\Big) ds \approx_d \|x\|_{\dot{W}_d^1}.$$

Comparison to the commutative case

In the commutative ($\theta = 0$) case then the formula for $\varphi(|\bar{d}x|^d)$ has been known since the 1980s (Connes. 1988)

$$\varphi(|\bar{d}f|^d) = c_d \int_{\mathbb{T}^d} \left(\sum_{j=1}^{d} |\partial_j f|^2 \right)^{d/2} dt.$$

It is insightful to explain how the noncommutative case reduces to the commutative case.

Comparison to the commutative case

Our formula is:

$$\varphi(|\bar{d}x|^d) = c_d\tau\Big(\int_{\mathbb{S}^{d-1}}\big(\sum_{j=1}^{d}|\partial_j x - s_j\sum_{k=1}^{d}s_k\partial_k x|^2\big)^{\frac{d}{2}}ds\Big).$$

If everything is commuting, we can take out a factor of:

$$\|\nabla x\|_2^d := \left(\sum_{j=1}^{d}|\partial_j x|^2\right)^{d/2}$$

and let $u_j = \frac{\partial_j x}{\|\nabla x\|_2}$ to get:

$$\varphi(|\bar{d}x|^d) = c_d\tau\left(\|\nabla x\|_2^d\int_{\mathbb{S}^{d-1}}\left(\sum_{j=1}^{d}|u_j - s_j\sum_{k=1}^{d}s_k u_k|^2\right)^{\frac{d}{2}}ds\right)$$

Comparison to the commutative case

Examine the inner integral:

$$\mathcal{I} := \int_{\mathbb{S}^{d-1}} \left(\sum_{j=1}^{d} |u_j - s_j \sum_{k=1}^{d} s_k u_k|^2 \right)^{\frac{d}{2}} ds.$$

Recall that $u_j = \frac{\partial_j x}{\|\nabla x\|_2}$ is a function on $\mathbb{T}^d$, so $\mathcal{I}$ is actually a function on $\mathbb{T}^d$.
For each $t \in \mathbb{T}^d$, we have:

$$\mathcal{I}(t) = \int_{\mathbb{S}^{d-1}} \left(\sum_{j=1}^{d} |u_j(t) - s_j \sum_{k=1}^{d} s_k u_k(t)|^2 \right)^{\frac{d}{2}} ds$$

Comparison to the commutative case

Now for each fixed t, $(u_1(t), u_2(t), \ldots, u_d(t))$ is a particular fixed unit vector in $\mathbb{R}^d$. Using the rotational invariance of the integral over $\mathbb{S}^{d-1}$, we can choose coordinates such that $(u_1(t), u_2(t), \ldots, u_d(t)) = (1, 0, \ldots, 0)$. In these coordinates, $\mathcal{I}(t)$ becomes:

$$\mathcal{I}(t) = \int_{\mathbb{S}^{d-1}} \left(\sum_{j=1}^{d} |\delta_{1,j} - s_j s_1|^2 \right)^{1/2} ds$$

so there is no dependence on t! Thus $\mathcal{I}$ is just a constant, and we recover:

$$\varphi(|\bar{d} f|^d) = k_d \int_{\mathbb{T}^d} \left(\sum_{j=1}^{d} |\partial_j f|^2 \right)^{d/2} dt.$$

where $k_d = c_d \mathcal{I}$.

Thank you!