Characterization of the decay structure for a dissipative linear system

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Key Words: Stability Condition, Regularity-loss structure

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1. Introduction

I. Introduction

Symmetric hyperbolic systems with relaxation:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{SHR}$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$.
Assume that

(a) A^0 is symmetric and positive definite,
(b) A^j is symmetric for each j,
(c) L is symmetric and non-negative definite.

Applying the Fourier transform, we obtan

$$A^0 \hat{u}_t + i|\xi|A(\omega)\hat{u} + L\hat{u} = 0,$$

where $A(\omega) = \sum_{j=1}^{n} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^{n-1}$.

Eigenvalue problem:

$$\lambda A^0\varphi + (i|\xi|A(\omega) + L)\varphi = 0.$$

$\lambda = \lambda(|\xi|, \omega)$: Eigenvalue, $\varphi = \varphi(|\xi|, \omega)$: Eigenvector.

Jin-Xin model:

$$\rho_t + v_x = 0,$$

$$v_t + \rho_x + v = 0.$$

We rewrite Jin-Xin model that

$$u_t + Au_x + Lu = 0,$$

where

$$u = \begin{pmatrix} \rho \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Stability conditions

Condition(CSC): Shizuta & Kawashima (1985)

For any $(\mu, \omega) \in \mathbb{R} \times S^{n-1}$, $\mathrm{Ker}(\mu I + A(\omega)) \cap \mathrm{Ker}(L) = \{0\}$.

Condition(CR): Kalman, Ho & Narendra (1963), Beauchard & Zuazua (2011)

For any $\omega \in S^{n-1}$,

$$\mathrm{rank} \begin{pmatrix} L \\ L(A^0)^{-1}A(\omega) \\ \vdots \\ L((A^0)^{-1}A(\omega))^{m-1} \end{pmatrix} = m.$$

Condition(CK): Umeda, Kawashima & Shizuta (1984)

There exists $K(\omega)$ with the following properties:

(i) $K(-\omega) = -K(\omega)$. (ii) $K(\omega)A^0$ is skew-symmetric.

(iii) $(K(\omega)A(\omega))^\sharp$ is positive definite on $\ker(L)$,

where $X^\sharp$ is the symmetric part of X.

Characterization and decay estimate

Theorem 1 (Characterization for the dissipative structure)

The following conditions are equivalent.

(i) Condition(CSC). (ii) Condition(CR). (iii) Condition(CK).

(iv) $\mathrm{Re}\lambda(|\xi|) < 0$ *for* $|\xi| \neq 0$. (v) $\mathrm{Re}\lambda(|\xi|) \leq -c|\xi|^2/(1+|\xi|^2)$.

Theorem 2 (Decay estimate)

Under Condition(CK)*, the solutions to* (SHR) *satisfy the pointwise estimate*

$$|\hat{u}(\xi,t)| \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|,$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$*. Namely, we obtain*

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}, \quad k \geq 0.$$

Q: Can we extend these conditions for (SHR) with non-symmetric L??

Examples

Dissipative Timoshenko system(linear):

$$\phi_{tt} - (\phi_x - \psi)_x = 0,$$
$$\psi_{tt} - \psi_{xx} - (\phi_x - \psi) + \psi_t = 0.$$

Putting $\rho = \phi_x - \psi$, $v = \phi_t$, $z = \psi_x$, $y = \psi_t$, we obtain the symmetric hyperbolic system

$$u_t + Au_x + Lu = 0,$$

where $u = (\rho, v, z, y)^T$ and

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Thermoelastic plate equation with Cattaneo's law:

$$\begin{cases} v_{tt} + \Delta^2 v + \Delta\theta = 0, \\ \theta_t + \operatorname{div} q - \Delta v_t = 0, \\ \tau q_t + q + \nabla\theta = 0. \end{cases}$$

Putting $z = \Delta u$ and $y = u_t$, then this yields

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} - \sum_{jk=1}^{n} B^{jk} u_{x_j x_k} + Lu = 0,$$

where $u = (z, y, \theta, q)^T$, $A^0 = \mathrm{diag}(1, 1, 1, \tau I)$, $L = \mathrm{diag}(0, 0, 0, I)$ and

$$A(\omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & \omega^T & 0 \end{pmatrix}, \quad B(\omega) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$(A(\omega) = \sum_{j=1}^n A^j \omega_j, \;\; B(\omega) = \sum_{jk=1}^n B^{jk} \omega_j \omega_k, \;\; \omega = \xi/|\xi| \in S^{n-1}.)$

2. New stability condition

Linear system:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} - \sum_{jk=1}^{n} B^{jk} u_{x_j x_k} + Lu = 0, \qquad \text{(LS)}$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$.

Apply the Fourier transform :

$$A^0 \hat{u}_t + i|\xi|A(\omega)\hat{u} + |\xi|^2 B(\omega)\hat{u} + L\hat{u} = 0,$$

where $A(\omega) = \sum_{j=1}^{n} A^j \omega_j$, $B(\omega) = \sum_{jk=1}^{n} B^{jk}\omega_j\omega_k$, $\omega = \xi/|\xi| \in S^{n-1}$.

Assume that

(a) A^0 is symmetric and positive definite,

(b) A^j is symmetric for each j,

(c) $B(\omega)^\sharp$ and $L^\sharp$ are non-negative definite (not necessary symmetric).

$X^\sharp$: symmetric part of X, $X^\flat$: skew-symmetric part of X

Eigenvalue problem:

$$\lambda A^0 \varphi + (irA(\omega) + r^2 B(\omega) + L)\varphi = 0. \tag{EP}$$

$\lambda = \lambda(r,\omega)$: Eigenvalue, $\varphi = \varphi(r,\omega)$: Eigenvector.

Remark 2.1

$\mathrm{Re}\lambda(r,\omega) \le 0$ for $r \ge 0$, $\omega \in S^{n-1}$

Indeed, taking a $\mathbb{C}^m$ inner product (EP) with φ, and taking a real part for the resultant equation, we obtain

$$\mathrm{Re}\lambda \langle A^0\varphi, \varphi\rangle + r^2 \langle B(\omega)^\sharp \varphi, \varphi\rangle + \langle L^\sharp \varphi, \varphi\rangle = 0. \tag{1}$$

Here, we used the symmetric property for $A(\omega)$. Therefore, since A^0 is positive definite, and $B(\omega)^\sharp$ and $L^\sharp$ are non-negative definite, we arrive at Remark 2.1.

New stability conditions

Define $\mathcal{A}(\nu,\omega) := (A^0)^{-1}(\nu A(\omega) - i\nu^2 B^\flat(\omega) - iL^\flat)$.

Here, $\nu A(\omega) - i\nu^2 B^\flat(\omega) - iL^\flat$ is a complex valued Hermitan matrix.

Stability Condition(SC):

For any $(\mu,\nu,\omega) \in \mathbb{R} \times \mathbb{R}_+ \times S^{n-1}$,

$$\mathrm{Ker}(\mu I + \mathcal{A}(\nu,\omega)) \cap \mathrm{Ker}(B^\sharp(\omega)) \cap \mathrm{Ker}(L^\sharp) = \{0\}$$

Kalman Rank Condition(R):

For any $(\nu,\omega) \in \mathbb{R}_+ \times S^{n-1}$,

$$\mathrm{rank}\begin{pmatrix} B(\omega)^\sharp \\ B(\omega)^\sharp \mathcal{A}(\nu,\omega) \\ \vdots \\ B(\omega)^\sharp \mathcal{A}(\nu,\omega)^{m-1} \\ L^\sharp \\ L^\sharp \mathcal{A}(\nu,\omega) \\ \vdots \\ L^\sharp \mathcal{A}(\nu,\omega)^{m-1} \end{pmatrix} = m.$$

New stability conditions

Craftsmanship Condition(K):

There exists $\mathcal{K}(\nu,\omega) \in C(\mathbb{R}_+ \times S^{n-1})$ with the following properties:

(i) $\bar{\mathcal{K}}(\nu,-\omega) = -\mathcal{K}(\nu,\omega)$. (ii) $\mathcal{K}(\nu,\omega)^* = -\mathcal{K}(\nu,\omega)$.

(iii) $\exists C_K$ s.t. $\|\mathcal{K}(\nu,\omega)\| \le C_K$ for $(\nu,\omega) \in \mathbb{R}_+ \times S^{n-1}$.

(iv) $\exists c_K$ s.t.

$$\langle (B(\omega)^\sharp + L^\sharp + (\mathcal{K}(\nu,\omega)\mathcal{A}(\nu,\omega))^\sharp)\sigma,\sigma\rangle > \frac{c_K\nu^2}{1+\nu^2}|\mathcal{K}(\nu,\omega)\sigma|^2$$

for $(\nu,\omega,\sigma) \in \mathbb{R}_+ \times S^{n-1} \times \mathbb{S}^{m-1}$, where $\mathbb{S}^{m-1} := \{\sigma \in \mathbb{C}^m; |\sigma| = 1\}$.

Characterization for the strict dissipativity

Theorem 3 (Characterization for the strict dissipativity,U(2019))

Let $n = 1$. *The following conditions are equivalent.*

(i) Condition(SC). (ii) Condition(R). (iii) Condition(K).

(iv) $\mathrm{Re}\lambda(r,\omega) < 0$ *for* $r > 0$, $\omega \in S^{n-1}$ (called Strictly dissipative).

(v)

$$\mathrm{Re}\lambda(r,\omega) \le \begin{cases} -c\dfrac{r^{2(m-1)}}{(1+r^2)^{2(m-1)}} & (A(\omega) \neq O,\ B(\omega) = O,\ L \neq O), \\ -c\dfrac{r^{2m}}{(1+r^2)^{2(m-2)}} & (A(\omega) \neq O,\ B(\omega) \neq O,\ L = O), \\ -c\dfrac{r^{2(2m-1)}}{(1+r^2)^{4m-3}} & (A(\omega) \neq O,\ B(\omega) \neq O,\ L \neq O) \end{cases}$$

for $\omega \in S^{n-1}$ (called Uniformly dissipative).

Characterization for the strict dissipativity

Theorem 4 (Characterization for the strict dissipativity, U(2018))

Let $n \geq 2$ *and* $B(\omega) = O$. *Suppose that* $\mathrm{Ker}\big(L^\sharp G\big(((A^0)^{-1}A(\omega))^\ell, (-i(A^0)^{-1}L^\flat)^{k-\ell}\big)\big)$ *does not depend on* $\omega \in S^{n-1}$. *The following conditions are equivalent.*

(i) Condition(SC). (ii) Condition(R). (iii) Condition(K).

(iv) $\mathrm{Re}\lambda(r,\omega) < 0$ *for* $r > 0$, $\omega \in S^{n-1}$.

(v) $\mathrm{Re}\lambda(r,\omega) \leq -cr^{2(m-1)}/(1+r^2)^{2(m-1)}$ *for* $\omega \in S^{n-1}$.

For matrices X and Y, we define $(X+Y)^k = \sum_{\ell=0}^k G(X^\ell, Y^{k-\ell})$, where $G(X^\ell, Y^{k-\ell})$ denotes a polynomial of X and Y, which degrees of X and Y are ℓ and $k-\ell$, respectively. Then $\mathcal{A}(r,\omega)^k$ is represented by

$$\begin{aligned}\mathcal{A}(r,\omega)^k &= \big((A^0)^{-1}(rA(\omega) - iL^\flat)\big)^k \\ &= \sum_{\ell=0}^{k} r^\ell G\big(((A^0)^{-1}A(\omega))^\ell, (-i(A^0)^{-1}L^\flat)^{k-\ell}\big).\end{aligned}$$

Corollary 5 (Decay estimate, U(2018))

Let $A(\omega) \neq 0,\ B(\omega) \neq O,\ L \neq O$. *Under the condition* (v) *in Theorem 3, the solutions to* (LS) *satisfy the pointwise estimate*

$$|\hat{u}(\xi,t)| \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \qquad \rho(\xi) = \frac{|\xi|^{2(2m-1)}}{(1+|\xi|^2)^{4m-3}}.$$

Namely, we obtain

$$\|u(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4(2m-1)}}\|u_0\|_{L^1} + C(1+t)^{-\frac{\ell}{4(m-1)}}\|\partial_x^\ell u_0\|_{L^2}, \quad \ell \geq 0.$$

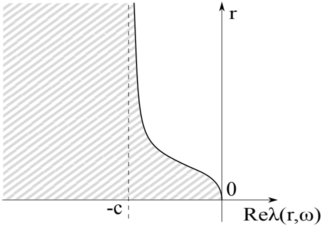


Figure: Standard type

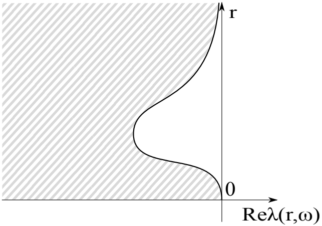


Figure: Regularity-loss type

Remarks

Remark 2.2

The pointwise estimate in Theorem 3 might not be optimal.
(e.g. Timoshenko system is $\mathrm{Re}\lambda(r,\omega) \le -cr^2/(1+r^2)^2$ but $m=4$.

Outline of the proof of Theorem 3:

♣ $\mathrm{Re}\lambda(r,\omega) < 0 \iff$ Condition(SC) $\iff$ Condition(R)

($\because$ Ccontradiction argument and the Celey-Hamilton theorem.)

♣ Condition(R) $\Longrightarrow$ Condition(K) $\Longrightarrow \mathrm{Re}\lambda(r,\omega) < 0$

($\because$ Use the energy method. Section 4.)

♣ Condition(R) $\Longrightarrow \mathrm{Re}\lambda(r,\omega) \le -c\dfrac{r^{2(2m-1)}}{(1+r^2)^{4m-3}}$

$$\Longrightarrow \mathrm{Re}\lambda(r,\omega) < 0$$

($\because$ Construct the Lyapunov function. Section 3.)

3. Lyapunov function

Lyapunov function

Let κ be a small positive number. Then we chose κ_k such that

$$0 = \kappa_0 < \kappa_1 < \cdots < \kappa_m, \qquad \kappa_k - \frac{1}{2}(\kappa_{k-1} + \kappa_{k+1}) \geq \kappa > 0. \tag{2}$$

Then we define the Lyapunov function

$$\mathcal{E}(\hat{u}) := \langle A^0 \hat{u}, \hat{u} \rangle + \delta h_1(|\xi|, \omega) \sum_{k=1}^{m-1} \varepsilon^{\kappa_k} \frac{\mathrm{Im}\langle L^\sharp \mathcal{A}(|\xi|,\omega)^{k-1}\hat{u}, L^\sharp \mathcal{A}(|\xi|,\omega)^k \hat{u}\rangle}{\|\mathcal{A}(|\xi|,\omega)\|^{2k}}$$

$$+ \delta h_2(|\xi|, \omega) \sum_{k=1}^{m-1} \varepsilon^{\kappa_k} \frac{\mathrm{Im}\langle B^\sharp(\omega) \mathcal{A}(|\xi|,\omega)^{k-1}\hat{u}, B^\sharp(\omega) \mathcal{A}(|\xi|,\omega)^k \hat{u}\rangle}{\|\mathcal{A}(|\xi|,\omega)\|^{2k}}$$

for $\delta > 0$ and $\varepsilon > 0$, where

$$h_1(|\xi|,\omega) := \frac{\|\mathcal{A}(|\xi|,\omega)\|^2}{(\|\mathcal{A}(|\xi|,\omega)\| + \|(A^0)^{-1}\|\|L^\sharp\|)^2},$$

$$h_2(|\xi|,\omega) := \frac{|\xi|^2\|\mathcal{A}(|\xi|,\omega)\|^2}{(\|\mathcal{A}(|\xi|,\omega)\| + |\xi|^2\|(A^0)^{-1}\|\|B^\sharp(\omega)\|)^2}.$$

Lyapunov function

Lemma 6 (Lyapunov function)

There exist δ_0 *and* ε_0 *such that*

$$\frac{\partial}{\partial t}\mathcal{E}(\hat{u}) + c_0|L^\sharp \hat{u}|^2 + c_1 h_1(|\xi|,\omega)\sum_{k=1}^{m-1}\varepsilon^{\kappa_k}\frac{|L^\sharp \mathcal{A}(|\xi|,\omega)^k\hat{u}|^2}{\|\mathcal{A}(|\xi|,\omega)\|^{2k}} + c_2 h_2(|\xi|,\omega)\sum_{k=1}^{m-1}\varepsilon^{\kappa_k}\frac{|B^\sharp(\omega)\mathcal{A}(|\xi|,\omega)^k\hat{u}|^2}{\|\mathcal{A}(|\xi|,\omega)\|^{2k}} \le 0, \tag{3}$$

and $c_*|\hat{u}|^2 \le \mathcal{E}(\hat{u}) \le C_*|\hat{u}|^2$ *for* $\delta = \delta_0$ *and* $0 < \varepsilon < \varepsilon_0$.

Remark 3.1

When we prove Lemma 6, we do not use the conditions in Theorem 3.

Energy estimate

Corollary 7

$$\frac{d}{dt}\mathcal{E}(\hat{u}) + c\mathcal{D}(|\xi|,\omega,\hat{u}) \le 0,$$

where

$$\mathcal{D}(|\xi|,\omega,\hat{u}) := \begin{cases} \displaystyle\sum_{k=0}^{m-1} \frac{|L^\sharp \mathcal{A}(|\xi|,\omega)^k \hat{u}|^2}{(1+|\xi|^2)^k} \\ \qquad \textit{if} \quad A(\omega) \ne O,\ B(\omega) = O,\ L^\sharp \ne O,\ L^\flat \ne O, \\ \displaystyle\sum_{k=0}^{m-1} \frac{|\xi|^2 |B^\sharp(\omega)\mathcal{A}(|\xi|,\omega)^k \hat{u}|^2}{|\xi|^{2k}(1+|\xi|^2)^k} \\ \qquad \textit{if} \quad A(\omega) \ne O,\ B^\sharp(\omega) \ne O,\ B^\flat(\omega) \ne O,\ L = O, \\ \displaystyle\sum_{k=0}^{m-1} \frac{|L^\sharp \mathcal{A}(|\xi|,\omega)^k \hat{u}|^2}{(1+|\xi|^2)^{2k}} + \sum_{k=0}^{m-1} \frac{|\xi|^2 |B^\sharp(\omega)\mathcal{A}(|\xi|,\omega)^k \hat{u}|^2}{(1+|\xi|^2)^{2k}} \\ \textit{if}\ A(\omega) \ne O,\ B^\sharp(\omega) \ne O, B^\flat(\omega) \ne O,\ L^\sharp \ne O, L^\flat \ne O. \end{cases}$$

Proof of Corollary 7

If $A(\omega) \neq O, B(\omega) = O, L^\sharp \neq O, L^\flat \neq O$, we obtain

$$\|\mathcal{A}(|\xi|,\omega)\| \sim (1+|\xi|), \qquad h_1(|\xi|,\omega) \sim 1.$$

If $A(\omega) \neq O, B^\sharp(\omega) \neq O, B^\flat(\omega) \neq O, L = O$, we obtain

$$\|\mathcal{A}(|\xi|,\omega)\| \sim |\xi|(1+|\xi|), \qquad h_2(|\xi|,\omega) \sim |\xi|^2.$$

If $A(\omega) \neq O, B^\sharp(\omega) \neq O, B^\flat(\omega) \neq O, L^\sharp \neq O, L^\flat \neq O$, we obtain

$$\|\mathcal{A}(|\xi|,\omega)\| \sim (1+|\xi|)^2, \qquad h_1(|\xi|,\omega) \sim 1, \qquad h_2(|\xi|,\omega) \sim |\xi|^2.$$

Proof of Theorem 3

Without loss of generality, we put $\omega = 1$. $(n = 1)$

$$L^\sharp \mathcal{A}(r,\omega)^k = L^\sharp\big((A^0)^{-1}(rA - ir^2B^\flat - iL^\flat)\big)^k = \sum_{\ell=0}^{2k} r^\ell G^1_{k,\ell},$$

$$B^\sharp(\omega)\mathcal{A}(r,\omega)^k = B^\sharp(\omega)\big((A^0)^{-1}(rA - ir^2B^\flat - iL^\flat)\big)^k = \sum_{\ell=0}^{2k} r^\ell G^2_{k,\ell},$$

where $G^1_{k,\ell}$ and $G^2_{k,\ell}$ are matrices constructed by $A^0, A(\omega), B(\omega), L$.
We define $N^j_{k,\ell} := \mathrm{Ker}\, G^j_{k,\ell}$ and

$$M^j_{k,\ell} := \min\{|G^j_{k,\ell}\sigma|^2 \ ; \ \sigma \in (N^j_{k,\ell})^\perp \cap \mathbb{S}^{m-1}\}, \quad M^j_0 := \min_{k,\ell} M^j_{k,\ell}$$

$$M_0 := \min\{M^1_0, M^2_0\}, \quad M_1 := \sum_{j=1}^{2}\sum_{k=0}^{m-1}\sum_{\ell=0}^{k} \max\{|G^j_{k,\ell}\sigma|^2 \ ; \ \sigma \in \mathbb{S}^{m-1}\}.$$

Remark that $M_0 > 0$ because of Condition (R).

In the low frequency region $0 < r \le r_0 := \min\{M_0/(12(m-1)M_1), 1\}$:

Because of Condition (R), for any r and σ, there is k_0 with $0 \le k_0 \le m-1$ such that $L^\sharp \mathcal{A}(r,\omega)^{k_0}\sigma \ne 0$ or $B^\sharp(\omega)\mathcal{A}(r,\omega)^{k_0}\sigma \ne 0$. Then, there exists a integer ℓ_0 with $0 \le \ell_0 \le 2k_0$ and j_0 with $j_0 = 1, 2$, satisfies
$G^{j_0}_{k_0,\ell}\sigma = 0$ for $0 \le \ell \le \ell_0 - 1$ and $G^{j_0}_{k_0,\ell_0}\sigma \ne 0$.

$$\mathcal{D}(r,\omega,\sigma) \ge \frac{1}{2^{2k_0}}|L^\sharp \mathcal{A}(r,\omega)^{k_0}\sigma|^2 + \frac{1}{2^{2k_0}} r^2 |B^\sharp(\omega)\mathcal{A}(r,\omega)^{k_0}\sigma|^2$$

$$\ge \frac{1}{2^{2k_0}}\Big|\sum_{\ell=\ell_0}^{2k_0} r^\ell G^1_{k_0,\ell}\sigma\Big|^2 + \frac{1}{2^{2k_0}}\Big|\sum_{\ell=\ell_0}^{2k_0} r^{\ell+1} G^2_{k_0,\ell}\sigma\Big|^2$$

$$\ge \frac{r^{2\ell_0}}{2^{2k_0+1}}(|G^1_{k_0,\ell_0}\sigma|^2 + r^2|G^2_{k_0,\ell_0}\sigma|^2) - \frac{3(k_0-\ell_0)}{2^{2k_0}}\sum_{\ell=\ell_0+1}^{2k_0} r^{2\ell}(|G^1_{k_0,\ell_0}\sigma|^2 + r^2|G^2_{k_0,\ell_0}\sigma|^2)$$

$$\ge \frac{r^{2(\ell_0+1)}}{2^{2k_0+1}} M_0 - \frac{3}{2^{k_0}}(m-1) r^{2(\ell_0+2)} M_1 \ge \frac{r^{2(\ell_0+1)}}{2^{2k_0+2}} M_0 \ge \frac{r^{4m-2}}{2^{2k_0+2}} M_0.$$

Proof of Lemma 6 ($B(\omega) = O$)

$$\begin{aligned}\tilde{\mathcal{E}}(\hat{u}) &:= \big(\|\mathcal{A}(|\xi|,\omega)\| + \|(A^0)^{-1}\|\|L^\sharp\|\big)^2 \|\mathcal{A}(|\xi|,\omega)\|^{2(m-2)}\mathcal{E}(\hat{u}) \\ &= \big(\|\mathcal{A}(|\xi|,\omega)\| + \|(A^0)^{-1}\|\|L^\sharp\|\big)^2 \|\mathcal{A}(|\xi|,\omega)\|^{2(m-2)}\langle A^0\hat{u},\hat{u}\rangle \\ &\quad + \delta\sum_{k=1}^{m-1}\varepsilon^{\kappa_k}a_k(|\xi|,\omega)^2\mathrm{Im}\langle L^\sharp\mathcal{A}(|\xi|,\omega)^{k-1}\hat{u}, L^\sharp\mathcal{A}(|\xi|,\omega)^k\hat{u}\rangle,\end{aligned}$$

where $a_k(|\xi|,\omega) := \|\mathcal{A}(|\xi|,\omega)\|^{m-k-1}$. Then we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\mathcal{E}}(\hat{u}) = -2\big(\|\mathcal{A}\| + \|L^\sharp\|\big)^2\|\mathcal{A}\|^{2(m-2)}\langle L^\sharp\hat{u},\hat{u}\rangle \\ -\delta\sum_{k=1}^{m-1}\varepsilon^{\kappa_k}a_k^2|L^\sharp\mathcal{A}^k\hat{u}|^2 - J_1 - J_2,\end{aligned}$$

where

$$J_1 := \delta\sum_{k=1}^{m-1}\varepsilon^{\kappa_k}a_k^2\mathrm{Re}\langle L^\sharp\mathcal{A}^{k-1}\hat{u}, L^\sharp\mathcal{A}^{k+1}\hat{u}\rangle,$$

$$\begin{aligned}J_2 := \delta\sum_{k=1}^{m-1}\varepsilon^{\kappa_k}a_k^2\big\{&\mathrm{Im}\langle L^\sharp\mathcal{A}^{k-1}(A^0)^{-1}L^\sharp\hat{u}, L^\sharp\mathcal{A}^k\hat{u}\rangle \\ &+ \mathrm{Im}\langle L^\sharp\mathcal{A}^{k-1}\hat{u}, L^\sharp\mathcal{A}^k(A^0)^{-1}L^\sharp\hat{u}\rangle\big\}.\end{aligned}$$

Proof of Lemma 6

Sinse $a_k^2 = a_{k-1}a_{k+1}$ and $\kappa_k - \frac{1}{2}(\kappa_{k-1} + \kappa_{k+1}) \geq \kappa > 0$, J_1 is estimated

$$
\begin{aligned}
|J_1| &\leq \delta \sum_{k=1}^{m-1} \varepsilon^{\kappa_k} a_{k-1}a_{k+1} |L^\sharp \mathcal{A}^{k-1}\hat{u}||L^\sharp \mathcal{A}^{k+1}\hat{u}| \\
&\leq \frac{\delta}{2}\varepsilon^{\kappa} \sum_{k=1}^{m-1} \left\{ \varepsilon^{\kappa_{k-1}} a_{k-1}^2 |L^\sharp \mathcal{A}^{k-1}\hat{u}|^2 + \varepsilon^{\kappa_{k+1}} a_{k+1}^2 |L^\sharp \mathcal{A}^{k+1}\hat{u}|^2 \right\} \\
&\leq \delta\varepsilon^{\kappa} \sum_{k=0}^{m-1} \varepsilon^{\kappa_k} a_k^2 |L^\sharp \mathcal{A}^k \hat{u}|^2 + \frac{\delta}{2}\varepsilon^{\kappa}\varepsilon^{\kappa_m} a_m^2 |L^\sharp \mathcal{A}^m \hat{u}|^2.
\end{aligned}
$$

By the Cayley-Hamilton theorem, we have $g(\tilde{\mathcal{A}}(|\xi|,\omega)) = O$, where

$$
g(\mu) := \det(\mu I - \tilde{\mathcal{A}}(|\xi|,\omega)) = \prod_{j=1}^{m} (\mu - \tau_j).
$$

Here $\tau_j = \tau_j(|\xi|,\omega) \in \mathbb{R}$ for $1 \leq j \leq m$ is an eigenvalue of the Hermitian matrix $\tilde{\mathcal{A}}(|\xi|,\omega) := (A^0)^{1/2}\mathcal{A}(|\xi|,\omega)(A^0)^{-1/2}$.

Therefore,

$$\tilde{\mathcal{A}}(|\xi|,\omega)^m=\sum_{k=0}^{m-1}\eta_k(|\xi|,\omega)\tilde{\mathcal{A}}(|\xi|,\omega)^k \Longrightarrow \mathcal{A}(|\xi|,\omega)^m=\sum_{k=0}^{m-1}\eta_k(|\xi|,\omega)\mathcal{A}(|\xi|,\omega)^k.$$

Here η_k is defined by τ_j for $1 \le j \le m$. Precisely, these are defined by

$$\eta_{m-1} := \sum_j \tau_j, \quad \eta_{m-2} := -\sum_{j_1<j_2} \tau_{j_1}\tau_{j_2}, \quad \eta_{m-3} := \sum_{j_1<j_2<j_3} \tau_{j_1}\tau_{j_2}\tau_{j_3}, \ \cdots$$

$$\eta_0 := (-1)^{m-1}\tau_1\tau_2\cdots\tau_m.$$

Remark that

$$\begin{aligned}|\eta_k(|\xi|,\omega)| &\le \binom{m}{m-k}\Big(\max_{1\le j\le m}|\tau_j(|\xi|,\omega)|\Big)^{m-k} = \binom{m}{m-k}\|\tilde{\mathcal{A}}(|\xi|,\omega)\|^{m-k} \\ &= \binom{m}{m-k}\|A^0\|^{(m-k)/2}\|(A^0)^{-1}\|^{(m-k)/2}\frac{a_k(|\xi|,\omega)}{a_m(|\xi|,\omega)}\end{aligned}$$

which comes from $\|\tilde{\mathcal{A}}(|\xi|,\omega)\| = \max_{1\le j\le m}|\tau_j(|\xi|,\omega)|$.

Proof of Lemma 6

Hence we have

$$
\begin{aligned}
a_m^2 |L^\sharp \mathcal{A}^m \hat{u}|^2 &\le a_m^2 \sum_{k=0}^{m-1} |\eta_k L^\sharp \mathcal{A}^k \hat{u}|^2 \\
&\le \sum_{k=0}^{m-1} \binom{m}{m-k}^2 \|A^0\|^{m-k} \|(A^0)^{-1}\|^{m-k} a_k^2 |L^\sharp \mathcal{A}^k \hat{u}|^2.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
|J_1| &\le \delta \varepsilon^\kappa \sum_{k=0}^{m-1} \varepsilon^{\kappa_k} a_k^2 |L^\sharp \mathcal{A}^k \hat{u}|^2 + \frac{\delta}{2} \varepsilon^\kappa \varepsilon^{\kappa_m} a_m^2 |L^\sharp \mathcal{A}^m \hat{u}|^2 \\
&\le \delta C_m \varepsilon^\kappa \sum_{k=0}^{m-1} \varepsilon^{\kappa_k} a_k^2 |L^\sharp \mathcal{A}^k \hat{u}|^2,
\end{aligned}
$$

where

$$
C_m := \max_{0 \le k \le m-1} \left\{ 1 + \frac{1}{2} \binom{m}{m-k}^2 \|A^0\|^{m-k} \|(A^0)^{-1}\|^{m-k} \right\}.
$$

Letting ε suitably small, we can control J_1. □

4. Craftsmanship condition

Condition(K) $\Longrightarrow \mathrm{Re}\lambda(|\xi|, \omega) < 0$

Craftsmanship Condition(K) ($B(\omega) = O$):

There exists $\mathcal{K}(\nu, \omega) \in C(\mathbb{R}_+ \times S^{n-1})$ with the following properties:

(i) $\bar{\mathcal{K}}(\nu, -\omega) = -\mathcal{K}(\nu, \omega)$. (ii) $\mathcal{K}(\nu, \omega)^* = -\mathcal{K}(\nu, \omega)$.

(iii) $\exists C_K$ s.t. $\|\mathcal{K}(\nu, \omega)\| \le C_K$ for $(\nu, \omega) \in \mathbb{R}_+ \times S^{n-1}$.

(iv) $\exists c_K$ s.t. $\langle (L^\sharp + (\mathcal{K}(\nu, \omega)\mathcal{A}(\nu, \omega))^\sharp)\sigma, \sigma\rangle > \frac{c_K \nu^2}{1+\nu^2}|\mathcal{K}(\nu, \omega)\sigma|^2$ for $(\nu, \omega, \sigma) \in \mathbb{R}_+ \times S^{n-1} \times \mathbb{S}^{m-1}$, where $\mathbb{S}^{m-1} := \{\sigma \in \mathbb{C}^m; |\sigma| = 1\}$.

Energy method: $\partial_t E + 2D = 0$,

$$E := \langle A^0 \hat{u}, \hat{u}\rangle - \frac{\gamma|\xi|^2}{1+|\xi|^2}\langle i\mathcal{K}\hat{u}, \hat{u}\rangle,$$

$$\begin{aligned} D &:= \langle L^\sharp \hat{u}, \hat{u}\rangle + \frac{\gamma|\xi|^2}{1+|\xi|^2}\langle (\mathcal{K}\mathcal{A})^\sharp \hat{u}, \hat{u}\rangle - \frac{\gamma|\xi|^2}{1+|\xi|^2}\mathrm{Re}(\langle i\mathcal{K}(A^0)^{-1}L^\sharp \hat{u}, \hat{u}\rangle) \\ &\ge \Big\{c_0 - \gamma C\Big(\frac{1}{4c_K} + \frac{|\xi|^2}{1+|\xi|^2}\Big)\Big\}|L^\sharp \hat{u}|^2 \\ &\quad + \frac{\gamma|\xi|^2}{(1+|\xi|^2)}\Big\{\langle (L^\sharp + (\mathcal{K}\mathcal{A})^\sharp)\hat{u}, \hat{u}\rangle - \frac{c_K|\xi|^2}{1+|\xi|^2}|\mathcal{K}\hat{u}|^2\Big\}. \end{aligned}$$

We introduce

$\mathcal{K}(r,\omega)$

$$:= \delta h(r,\omega) \sum_{k=1}^{m-1} \frac{\varepsilon^{\kappa_k} \{\mathcal{A}(r,\omega)^k (L^\sharp)^2 \mathcal{A}(r,\omega)^{k-1} - \mathcal{A}(r,\omega)^{k-1} (L^\sharp)^2 \mathcal{A}(r,\omega)^k\}}{\|\mathcal{A}(r,\omega)\|^{2k}}$$

Then we obtain

$$\langle (L^\sharp + (\mathcal{K}(r,\omega)\mathcal{A}(r,\omega))^\sharp)\sigma, \sigma\rangle - \frac{c_K r^2}{1+r^2} |\mathcal{K}(r,\omega)\sigma|^2$$
$$\geq (c_0 - \delta h(r,\omega))|L^\sharp \sigma|^2$$
$$+ \Big(1 - \varepsilon^\kappa C_m - \frac{4m\delta c_K r^2 h(r,\omega)\|L^\sharp\|^2}{(1+r^2)\|\mathcal{A}(\nu,\omega)\|^2}\Big) \delta h(r,\omega) \sum_{k=0}^{m-1} \varepsilon^{\kappa_k} \frac{|L^\sharp \mathcal{A}(\nu,\omega)^k \sigma|^2}{\|\mathcal{A}(\nu,\omega)\|^{2k}}.$$

$$h(r,\omega) := \frac{\|\mathcal{A}(r,\omega)\|^2}{(\|\mathcal{A}(r,\omega)\| + \|(A^0)^{-1}\|\|L^\sharp\|)^2}.$$

5. Application and weak dissipative structure

Applications

Cond.(SC): $\mu A^0\varphi + (\nu A(\omega) - iL^\flat)\varphi = 0, \quad \varphi \in \mathrm{Ker}(L^\sharp) \implies \varphi = 0$

Dissipative Timoshenko system:

$$A^0 = I, \quad A(\omega) = -\omega \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Corollary 8

The Timoshenko system satisfies Condition(SC)*. Namely, this system is strictly dissipative.*

Proof: For $(\mu, \nu, \omega) \in \mathbb{R} \times \mathbb{R}_+ \times \{-1, 1\}$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in \mathbb{C}^4$,

$$\begin{cases} \mu\varphi_1 - \nu\omega\varphi_2 - i\varphi_4 = 0, \\ \mu\varphi_2 - \nu\omega\varphi_1 = 0, \\ \mu\varphi_3 - \nu\omega\varphi_4 = 0, \\ \mu\varphi_4 - \nu\omega\varphi_3 + i\varphi_1 = 0, \end{cases} \quad \text{and} \quad \varphi_4 = 0. \quad \implies \quad \varphi = 0.$$

□

Thermoelastic plate equation with Cattaneo's law:

$$\begin{cases} v_{tt} + \Delta^2 v + \Delta\theta = 0, \\ \theta_t + \operatorname{div} q - \Delta v_t = 0, \\ \tau q_t + q + \nabla\theta = 0. \end{cases}$$

Putting $z = \Delta u$ and $y = u_t$, then this yields $u = (z, y, \theta, q)^T$, $A^0 = \mathrm{diag}(1, 1, 1, \tau I)$, $L = \mathrm{diag}(0, 0, 0, I)$ and

$$A(\omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & \omega^T & 0 \end{pmatrix}, \quad B(\omega) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Corollary 9

The thermoelastic plate equation with Cattaneo's law satisfies $\mathrm{Condition(SC)}$. *Namely, this system is strictly dissipative.*

♣ We had already obtained the sharp decay estimate(Racke-U(2016)).

Dissipative Bresse system: $(\ell \neq 0)$

$$\begin{aligned} \phi_{tt} - (\phi_x + \psi + \ell w)_x - \ell(w_x - \ell\phi) &= 0, \\ \psi_{tt} - \psi_{xx} + (\phi_x + \psi + \ell w) + \psi_t &= 0, \\ w_{tt} - (w_x - \ell\phi)_x + \ell(\phi_x + \psi + \ell w) &= 0. \end{aligned}$$

♣ If $\ell = 0$, this system is reduced to the dissipative Timoshenko system.

Putting $\rho = \phi_x + \psi + \ell w$, $v = \phi_t$, $z = \psi_x$, $y = \psi_t$, $q = w_x - \ell\phi$, $p = w_t$, we obtain the symmetric hyperbolic system $u_t + Au_x + Lu = 0$, where $u = (\rho, v, z, y, q, p)^T$ and

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -\ell \\ 0 & 0 & 0 & 0 & -\ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \ell & 0 & 0 & 0 & 0 \\ \ell & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

New dissipative structure

Corollary 10

The Bresse system does not satisfy Condition(SC). *Namely, this system is NOT strictly dissipative.*

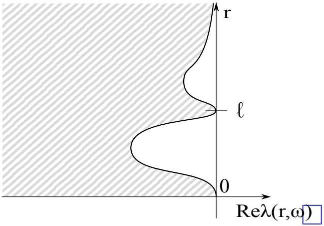
Proof: For $(\mu, \nu, \omega) \in \mathbb{R} \times \mathbb{R}_+ \times \{-1, 1\}$ and $\varphi = (\varphi_1, \cdots, \varphi_6)^T \in \mathbb{C}^6$,

$$\begin{cases} \mu\varphi_1 - \nu\omega\varphi_2 + i\varphi_4 + i\ell\varphi_6 = 0, \\ \mu\varphi_2 - \nu\omega\varphi_1 + i\ell\varphi_5 = 0, \\ \mu\varphi_3 - \nu\omega\varphi_4 = 0, \\ \mu\varphi_4 - \nu\omega\varphi_3 - i\varphi_1 = 0, \\ \mu\varphi_5 - \nu\omega\varphi_6 - i\ell\varphi_2 = 0, \\ \mu\varphi_6 - \nu\omega\varphi_5 - i\ell\varphi_1 = 0, \end{cases} \quad \text{and} \quad \varphi_4 = 0. \tag{4}$$

Let $(\mu, \nu) = (0, |\ell|)$, then

$$\varphi = (\sigma_1, \sigma_2, -i\frac{1}{|\ell|}\sigma_1, 0, -i\frac{\ell}{|\ell|}\sigma_1, -i\frac{\ell}{|\ell|}\sigma_2)^T$$

satisfy (4) for $(\sigma_1, \sigma_2) \in \mathbb{C}^2$.



Future Works

- Can we classify the estimate of eigenvalues for our system under Condition(SC)??

  ♣ Duan-Kawashima-U(2012) :

  Condition(S) + (K) $\Longrightarrow \mathrm{Re}\lambda \le -c\dfrac{|\xi|^2}{(1+|\xi|^2)^2}$

  (e.g. Timoshenko system, Euler-Maxwell system)

  ♣ Duan-Kawashima-U(2017) :

  Craftsmanship Condition $\Longrightarrow \mathrm{Re}\lambda \le -c\dfrac{|\xi|^4}{(1+|\xi|^2)^3}$

  (e.g. Timoshenko system with memory)

- Can we get the decay estimate for the dissipative Bresse system??

- How about the asymptotic profile of the solution to the dissipative Bresse system??

Thank You Very Much
for Your Attention!!