Width of semiclassical resonances above an energy level crossing

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joint works with

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1 Assumptions and Result

Let P be a 2×2 matrix-valued one-dimensional Schrödinger operator,

$$P = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix},$$

where $h > 0$ is a small parameter and

$$P_j = h^2 D_x^2 + V_j(x), \qquad (D_x = \frac{1}{i}\frac{d}{dx}),$$
$$W = W(x, hD_x) = r_0(x) + i r_1(x) hD_x$$

We suppose that $V_1(x)$ and $V_2(x)$ cross each other at one point and study the asymptotic distribution as $h \to 0$ of resonances near a fixed energy $E_0 \in \mathbb{R}$ above the crossing level.

1.1 Assumptions on V_1, V_2, E_0 and W

(A1) $V_1(x)$, $V_2(x)$ are real-valued on $\mathbb{R}$ and analytic in

$$\mathcal{S} := \{x \in \mathbb{C}\,;\ |\mathrm{Im}\, x| < \delta_0 \langle \mathrm{Re}\, x \rangle\}.$$

(A2) $V_1(x)$, $V_2(x)$ admit limits $V_1^{\pm}$, $V_2^{\pm}$ as Re $x \to \pm\infty$ in $\mathcal{S}$, and

$$V_2^{+} < 0 < E_0 < \min(V_1^{\pm}, V_2^{-}).$$

(A3) $\exists a < \exists b < 0 < \exists c$, s.t.

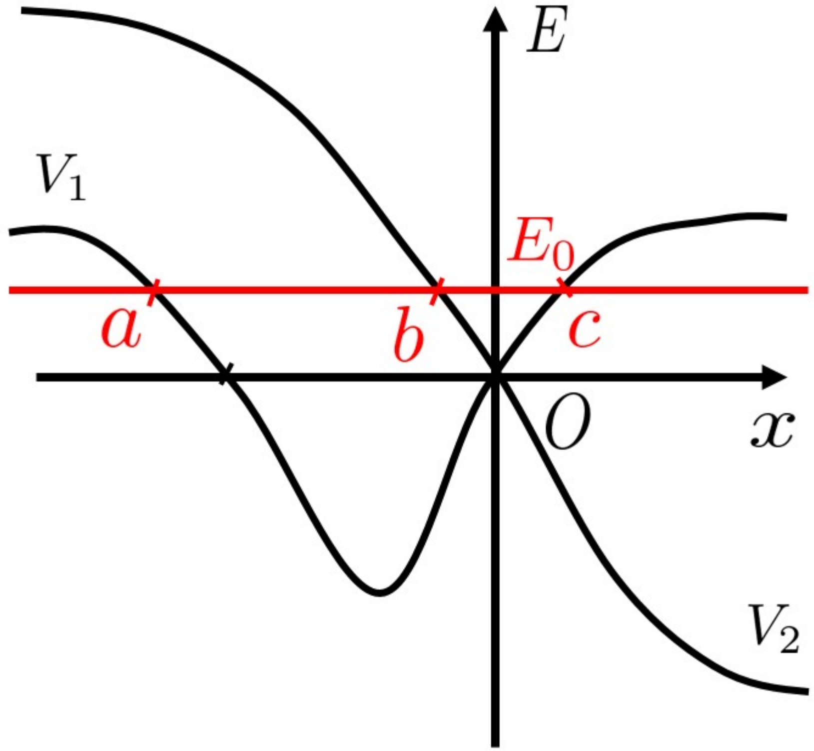
$$V_1 > E_0 \text{ on } (-\infty, a) \cup (c, +\infty),$$

$$V_1 < E_0 \text{ on } (a, c),$$

$$V_2 > E_0 \text{ on } (-\infty, b),$$

$$V_2 < E_0 \text{ on } (b, +\infty),$$

$$V_1'(a) < 0, \quad V_1'(c) > 0, \quad V_2'(b) < 0.$$



(A4) $V_1(x)$ and $V_2(x)$ cross each other at one point $x = 0$

$$V_1(0) = V_2(0) = 0,$$

$$V_1'(0) > 0, \quad V_2'(0) < 0.$$

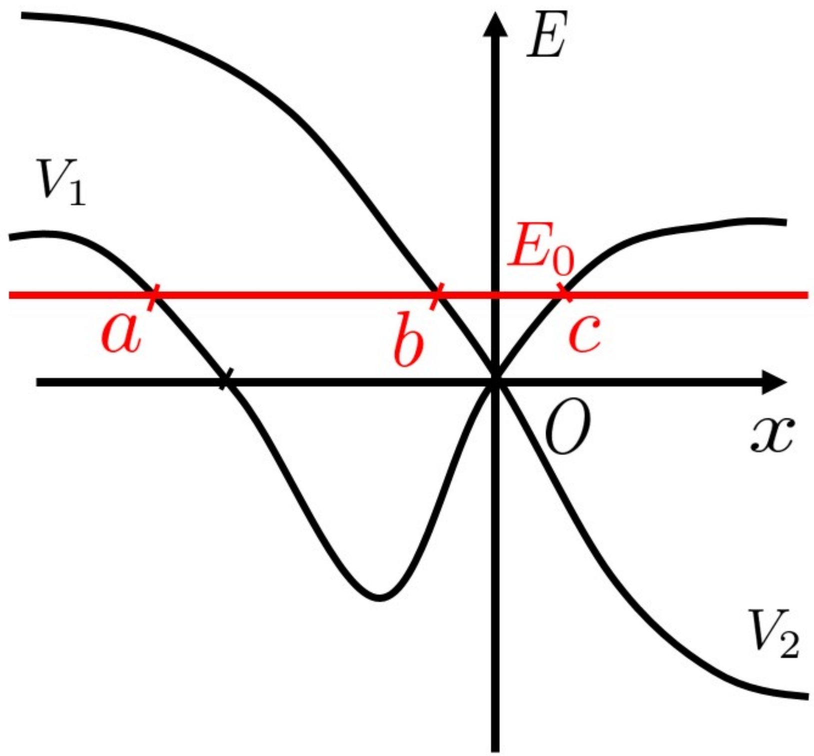
(A5) r_0, r_1 are bounded analytic functions on $\mathcal{S}$,

real on $\mathbb{R}$, and satisfy

$$(r_0(0), r_1(0)) \neq (0, 0).$$

Note that

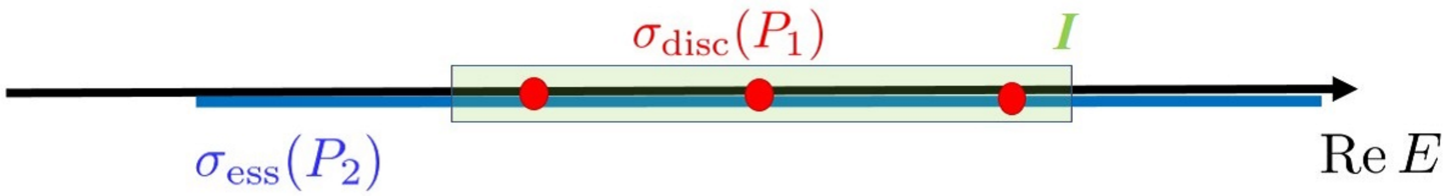
- **(A5)** means the ellipticity condition at $(0, \pm\sqrt{E_0})$ for $W = r_0(x) + ir_1(x)hD_x$.



Remark If $W \equiv 0$ (which is not the case here), then we know

$$\sigma(P) \cap I = (\sigma_{\text{disc}}(P_1) \cup \sigma_{\text{ess}}(P_2)) \cap I,$$

for a small interval $I \subset \mathbb{R}$ near E_0.

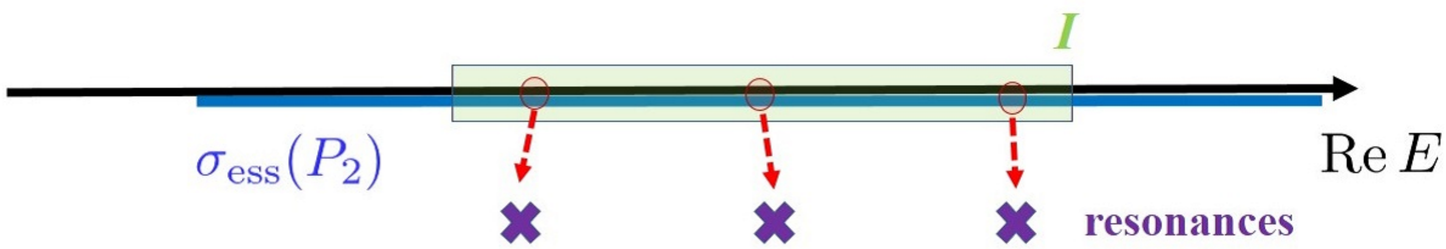


Remark If $\boldsymbol{W} \equiv \mathbf{0}$ (which is not the case here), then we know

$$\boldsymbol{\sigma}(\boldsymbol{P}) \cap \boldsymbol{I} = (\boldsymbol{\sigma}_{\text{disc}}(\boldsymbol{P_1}) \cup \boldsymbol{\sigma}_{\text{ess}}(\boldsymbol{P_2})) \cap \boldsymbol{I},$$

for a small interval $\boldsymbol{I} \subset \mathbb{R}$ near $\boldsymbol{E_0}$.

If $\boldsymbol{W} \not\equiv \mathbf{0}$, the perturbations (i.e. $\boldsymbol{W}, \boldsymbol{W}^*$) may create resonances near $\boldsymbol{\sigma}_{\text{disc}}(\boldsymbol{P_1})$.



We expect to find resonances instead in the complex nbd of $\boldsymbol{\sigma}_{\text{disc}}(\boldsymbol{P_1})$.

Aim We want to study the asymptotic expansion of resonances with respect to $\boldsymbol{h}$ small enough, in particular the imaginary part of the resonance (width of resonance).

1.2 Resonances

A (quantum) resonance is defined by

$$E \in \mathbb{C}_- : \text{resonance} \;\Leftrightarrow\; \exists u \neq 0 \text{ s.t. } \begin{cases} Pu = Eu, \\ u : \text{"outgoing"} \end{cases}$$

$$u : \text{outgoing} \Leftrightarrow u(xe^{i\theta}) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \text{ for small } \theta > 0.$$

In the 4 dimensional solution space, when $\exists 2$ solutions in $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-)$ and $\exists 2$ solutions in $L^2(\mathbb{R}_+ e^{i\theta}) \oplus L^2(\mathbb{R}_+ e^{i\theta})$, and then

E is a resonance iff these 4 solutions are linearly dependent.

We denote the set of resonances by $\mathrm{Res}\,(P)$.

For $E \in \mathrm{Res}\,(P)$, $(\mathrm{Im} E)^{-1}$ is considered to be the life time of the quantum state.

1.3 Results

Let $I_0 \subset \mathbb{R}$ be an interval including E_0 such that the function $\mathcal{A} : I_0 \mapsto \mathcal{A}(I_0)$ given by

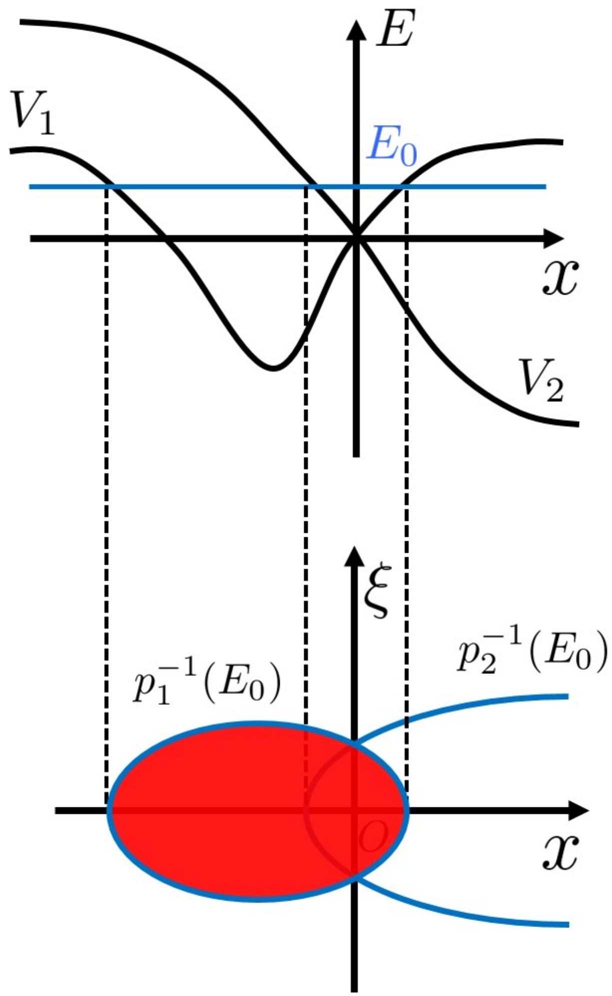
$$\mathcal{A}(E) = \int_{a(E)}^{c(E)} \sqrt{E - V_1(t)}dt$$

is strictly increasing. Here $a(E)$ (resp. $c(E)$) is the unique root of $V_1 = E$ close to a (resp. c). For $k \in \mathbb{Z}$ with $(k + \frac{1}{2})\pi h \in \mathcal{A}(I_0)$, we set

$$e_k(h) := \mathcal{A}^{-1}((k + \frac{1}{2})\pi h).$$

Note that the quantity $\mathcal{A}(E_0)$ stands for the area bounded by the characteristic set $p_1^{-1}(E_0)$.

We fix a small $\delta_0 > 0$ and an arbitrarily large $C_0 > 0$, and let $\mathcal{D}_h(\delta_0, C_0) = [E_0 - \delta_0, E_0 + \delta_0] - i[0, C_0 h]$.



Theorem For h small enough, we have

$$\mathrm{Res}(P) \cap \mathcal{D}_h(\delta_0, C_0) = \{E_k(h); k \in \mathbb{Z}\} \cap \mathcal{D}_h(\delta_0, C_0)$$

with complex numbers $E_k(h)$'s satisfying

$$\text{(i)} \qquad \mathrm{Re}\, E_k(h) = e_k(h) + \mathcal{O}(h^2),$$

$$\text{(ii)} \qquad \mathrm{Im}\, E_k(h) = -C(e_k(h))h^2 + \mathcal{O}(h^{7/3}),$$

where

$$C(E) = \frac{\pi}{\gamma \mathcal{A}'(E)} \left| r_0(0) E^{-\frac{1}{4}} \sin\left(\frac{\mathcal{B}(E)}{h} + \frac{\pi}{4}\right) + r_1(0) E^{\frac{1}{4}} \cos\left(\frac{\mathcal{B}(E)}{h} + \frac{\pi}{4}\right) \right|^2$$

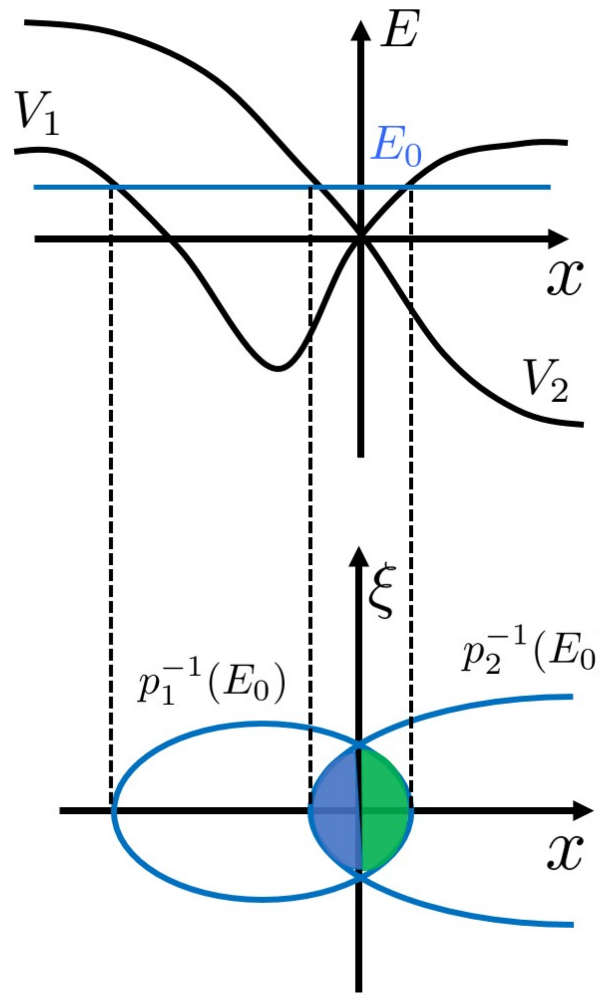
with $\gamma := V_1'(0) - V_2'(0) > 0$,

$$\mathcal{B}(E) := \int_{b(E)}^{0} \sqrt{E - V_2(x)}\,dx + \int_0^{c(E)} \sqrt{E - V_1(x)}\,dx,$$

where $b(E)$ is the unique root of $V_2(x) = E$ close to b.

Note that the quantity $\boldsymbol{\mathcal{B}(E_0)}$ stands for the area bounded by the characteristic sets $\boldsymbol{p_1^{-1}(E_0)}$ and $\boldsymbol{p_2^{-1}(E_0)}$.

$$\boldsymbol{\mathcal{B}(E) := \int_{b(E)}^{0} \sqrt{E - V_2(x)}\,dx + \int_{0}^{c(E)} \sqrt{E - V_1(x)}\,dx.}$$



1.4 Related works

We focus on the imaginary part of resonances.

- Exp. decay (Tunneling effect)
  - Non-crossing case $V_1 = x^2$, $V_2 = -x - 1$, $E_0 = 0$ (Martinez 1994, Nakamura 1995, Baklouti 1998)
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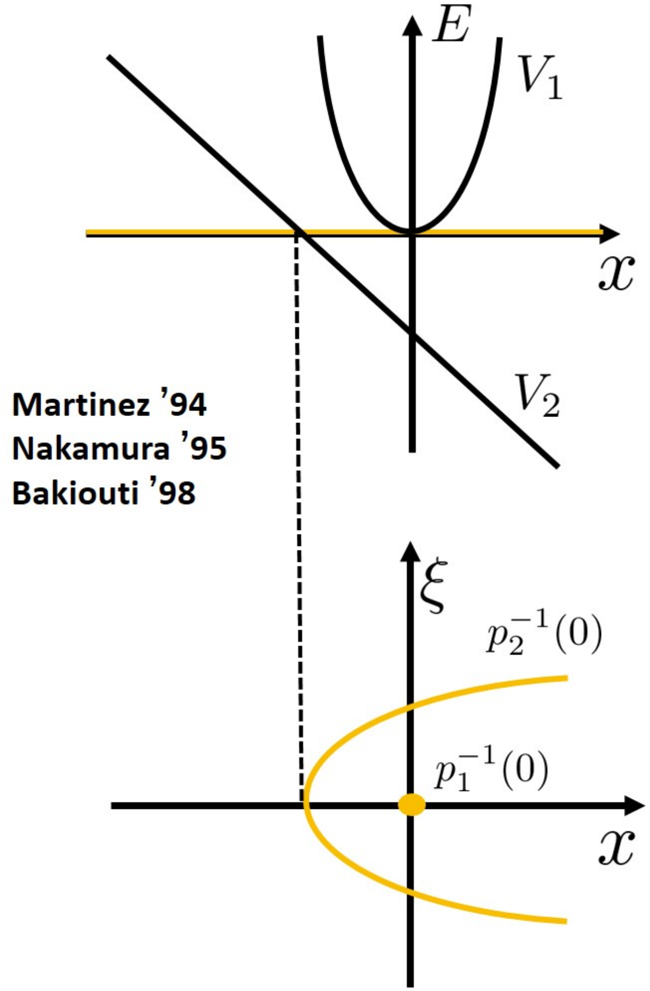

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  - Near the bottom of the well $\boldsymbol{V_1}$ (Grigis-Martinez 2014)
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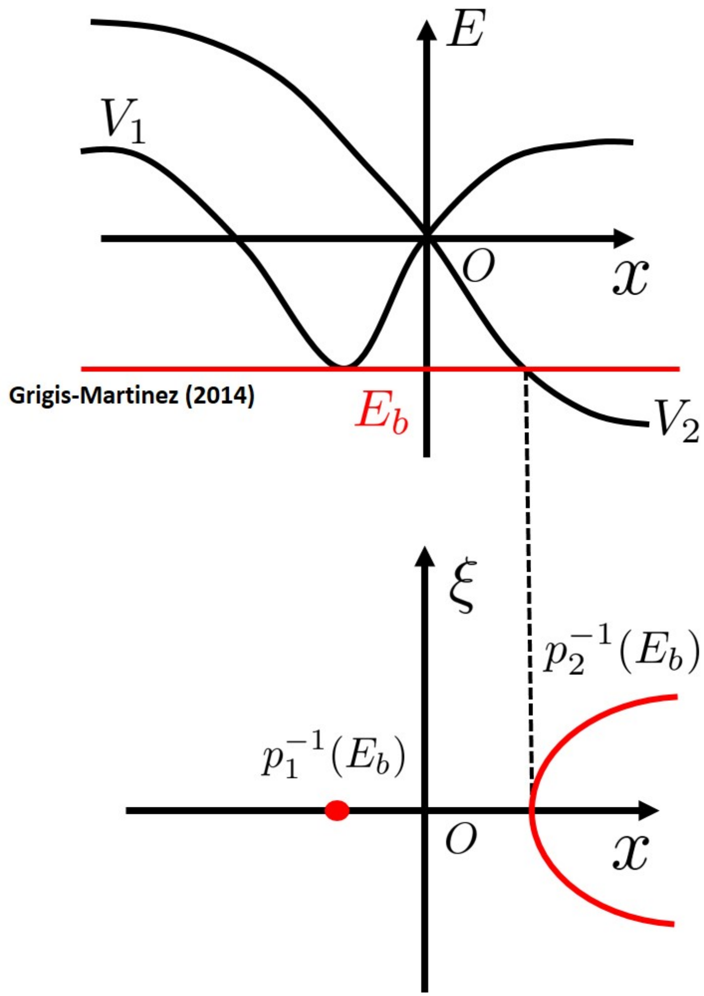

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  - Below the crossing (Ashida 2018)

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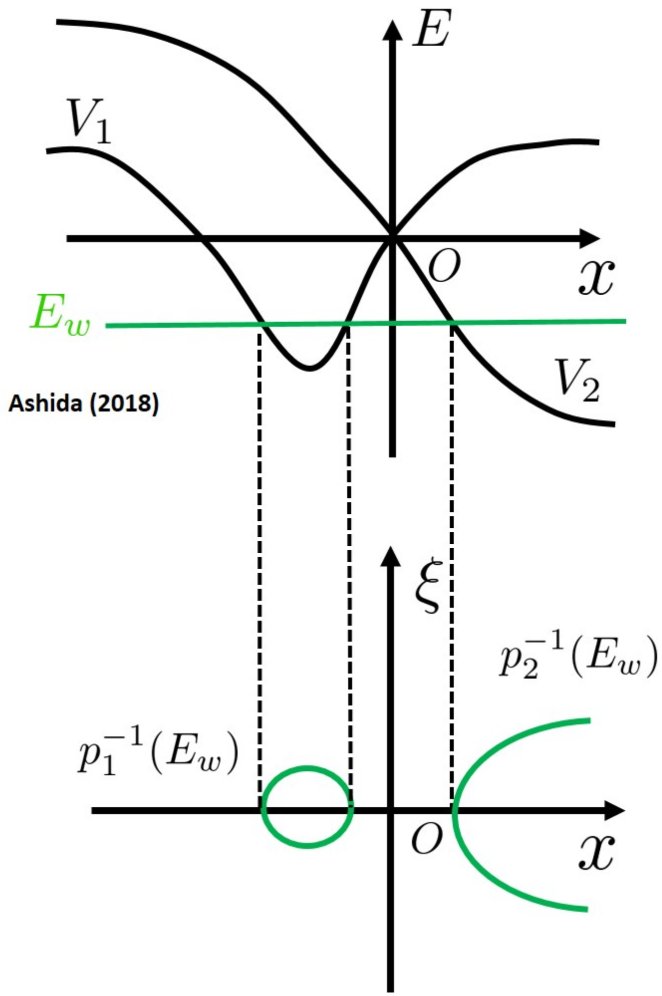

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  - Below the crossing (Ashida 2018)
- Polynomial decay
  - Near the crossing
    - ∗ elliptic interaction (F-M-W 2016); $h^{5/3}$
    - ∗ vector field interaction $r_0 \equiv 0$ (F-M-W 2017); $h^{7/3}$
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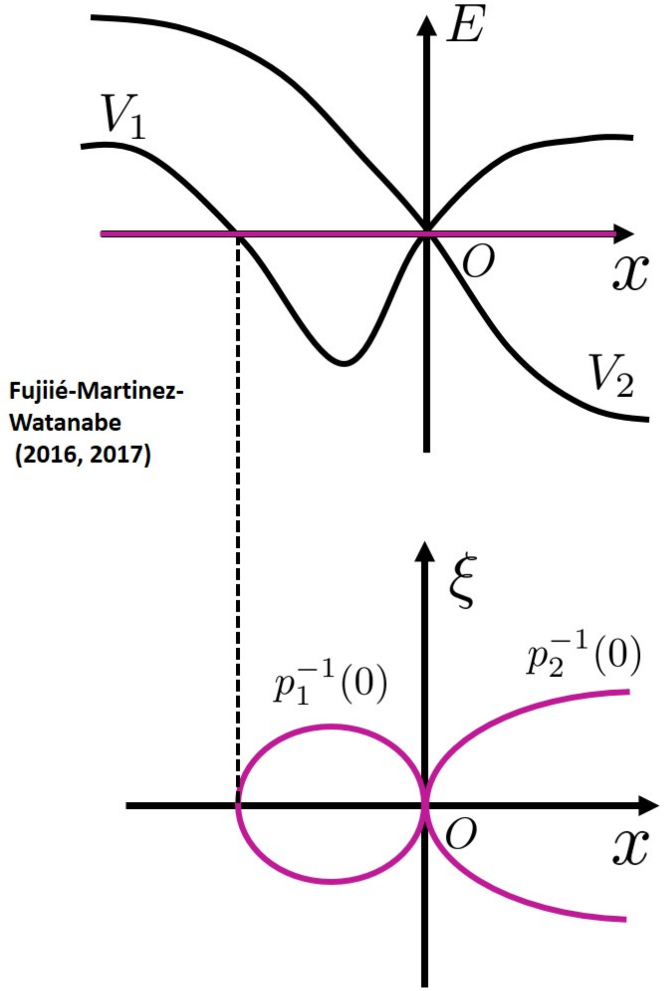

1.8 Related works

We focus on the imaginary part of resonances.

- Exp. decay (Tunneling effect)
  - Non-crossing case $V_1 = x^2$, $V_2 = -x - 1$, $E_0 = 0$ (Martinez 1994, Nakamura 1995, Baklouti 1998)
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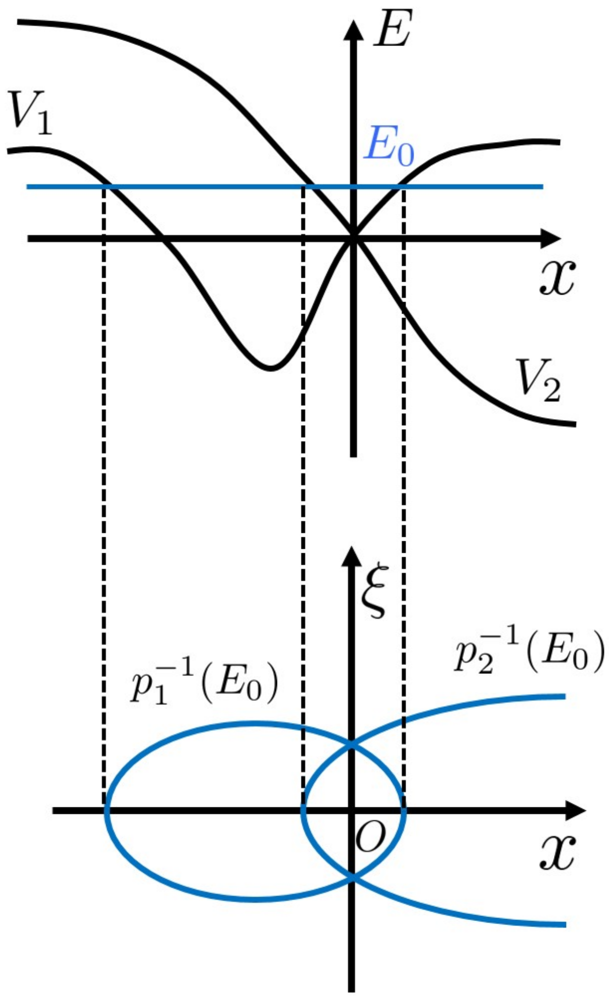
    - vector field interaction $r_0 \equiv 0$ (F-M-W 2017); $h^{7/3}$
  - Above the crossing h^2



In our previous works (FMW 2016, 2017), we characterized the semiclassical distribution of resonances near a crossing level ($E_0 = 0$) by means of $\lambda_k(h)h^{2/3}$ instead of $e_k(h)$, and we obtained, under the simple setting $V_1'(0) = -V_2'(0) = 1$,

$$\text{Im}\, E_k = -\frac{\pi^2 r_0(0)^2}{2^{\frac{2}{3}}\mathcal{A}'(0)}\left(\mathbf{Ai}(-2^{\frac{2}{3}}\lambda_k)\right)^2 h^{\frac{5}{3}} + \mathcal{O}(h^2) \qquad \textbf{(FMW1)}$$

$$\text{Im}\, E_k = -\frac{\pi^2 r_1(0)^2}{2^{\frac{4}{3}}\mathcal{A}'(0)}\left(\mathbf{Ai}'(-2^{\frac{2}{3}}\lambda_k)\right)^2 h^{\frac{7}{3}} + \mathcal{O}(h^{\frac{8}{3}}) \qquad \textbf{(FMW2)} : r_0(x) \equiv 0$$

where $\mathbf{Ai}$ stands for the Airy function $\mathbf{Ai}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[ix\xi + \xi^3/3\right] d\xi$.

From the asymptotic behavior of $\mathbf{Ai}(x)$ as $x \to -\infty$, one sees if $\lambda_k \to \infty$

$$\left(\mathbf{Ai}(-2^{\frac{2}{3}}\lambda_k)\right)^2 \sim \frac{1}{\pi} 2^{-\frac{1}{3}} \lambda_k^{-\frac{1}{2}} \sin^2\left(\frac{4}{3}\lambda_k^{3/2} + \frac{\pi}{4}\right)$$

$$\left(\mathbf{Ai}'(-2^{\frac{2}{3}}\lambda_k)\right)^2 \sim \frac{1}{\pi} 2^{+\frac{1}{3}} \lambda_k^{+\frac{1}{2}} \cos^2\left(\frac{4}{3}\lambda_k^{3/2} + \frac{\pi}{4}\right)$$

Regarding $e_k(h)$ as $\lambda_k h^{2/3}$ (i.e. crossing level $E_0 = 0$), we can confirm the matching the main result and our previous results. By using the asymptotic formula of Airy function, we have if $\lambda_k \to \infty$

$$\text{Im}\, E_k \sim -\frac{\pi r_0(0)^2}{2\mathcal{A}'(0)}\lambda_k^{-\frac{1}{2}} h^{\frac{5}{3}} \sin^2\left(\frac{4}{3}\lambda_k^{\frac{3}{2}} + \frac{\pi}{4}\right) \qquad \text{(FMW1)}$$

$$\text{Im}\, E_k \sim -\frac{\pi r_1(0)^2}{2\mathcal{A}'(0)}\lambda_k^{\frac{1}{2}} h^{\frac{7}{3}} \cos^2\left(\frac{4}{3}\lambda_k^{\frac{3}{2}} + \frac{\pi}{4}\right) \qquad \text{(FMW2)}$$

On the contrary, in this case, $\gamma = 2$, $\mathcal{A}'(\lambda_k h^{\frac{2}{3}}) \sim \mathcal{A}'(0)$ and $\mathcal{B}(\lambda_k h^{\frac{2}{3}}) \sim \frac{4}{3}\lambda_k^{\frac{3}{2}} h$, so that the main result, that is Im E_k behaves like

$$-\frac{\pi h^2}{\gamma \mathcal{A}'(E)}\left| r_0(0) E^{-\frac{1}{4}} \sin\left(\frac{\mathcal{B}(E)}{h} + \frac{\pi}{4}\right) + r_1(0) E^{\frac{1}{4}} \cos\left(\frac{\mathcal{B}(E)}{h} + \frac{\pi}{4}\right)\right|^2$$

with $E = \lambda_k h^{\frac{2}{3}}$, reproduces the above formulae of the previous results.

2 Sketch of proof

1st step Construction of resonant states

- Constructing out-going solutions (resonant states), i.e. checking the existence of them
- Computing the wronskian of them and deriving a rough estimate of resonances

2nd step Microlocal approach for precise asymptotics

- Connecting along the characteristic sets a normalized microlocal WKB solution which vanishes on the characteristic set corresponding to the in-coming trajectory
- Studying the connection formulae over the crossing point and the turning points
  - Through the crossing point, applying the reduction to a single-valued normal form (Helffer-Sjöstrand)
  - Through the turning point, applying the Maslov theory
- The connection along the closed trajectory refines the rough estimate obtained in the 1st step
- The connection toward the out-going trajectory gives the precise estimate of the imaginary part

2.1 Construction of resonant states

1. Construct 2 pairs of independent outgoing solutions $w_{1,L}, w_{2,L} \in L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-)$ and $w_{1,R}, w_{2,R} \in L^2(\mathbb{R}_+ e^{i\theta}) \oplus L^2(\mathbb{R}_+ e^{i\theta})$.

2. Compute the wronskian $\mathcal{W}(E,h)$ of these solutions:

$$\mathcal{W}(E,h) = \cos\frac{\mathcal{A}(E)}{h} + \mathcal{O}(h^{1/6}).$$

3. Deduce a rough estimate of resonances from the quantization condition $\mathcal{W}(E,h) = 0$:

$$E_k(h) = e_k(h) + \mathcal{O}(h^{7/6}).$$

The method of this part is essentially same as that of our previous work, which reduces the problem of the system to that of the single-valued problem of studying the fundamental solutions $(P_j - E)^{-1}$.

2.2 Microlocal approach for precise asymptotics

Let $u \in L^2$ and (x_0, ξ_0) be a point of the phase space $\mathbb{R}_x \times \mathbb{R}_\xi$.

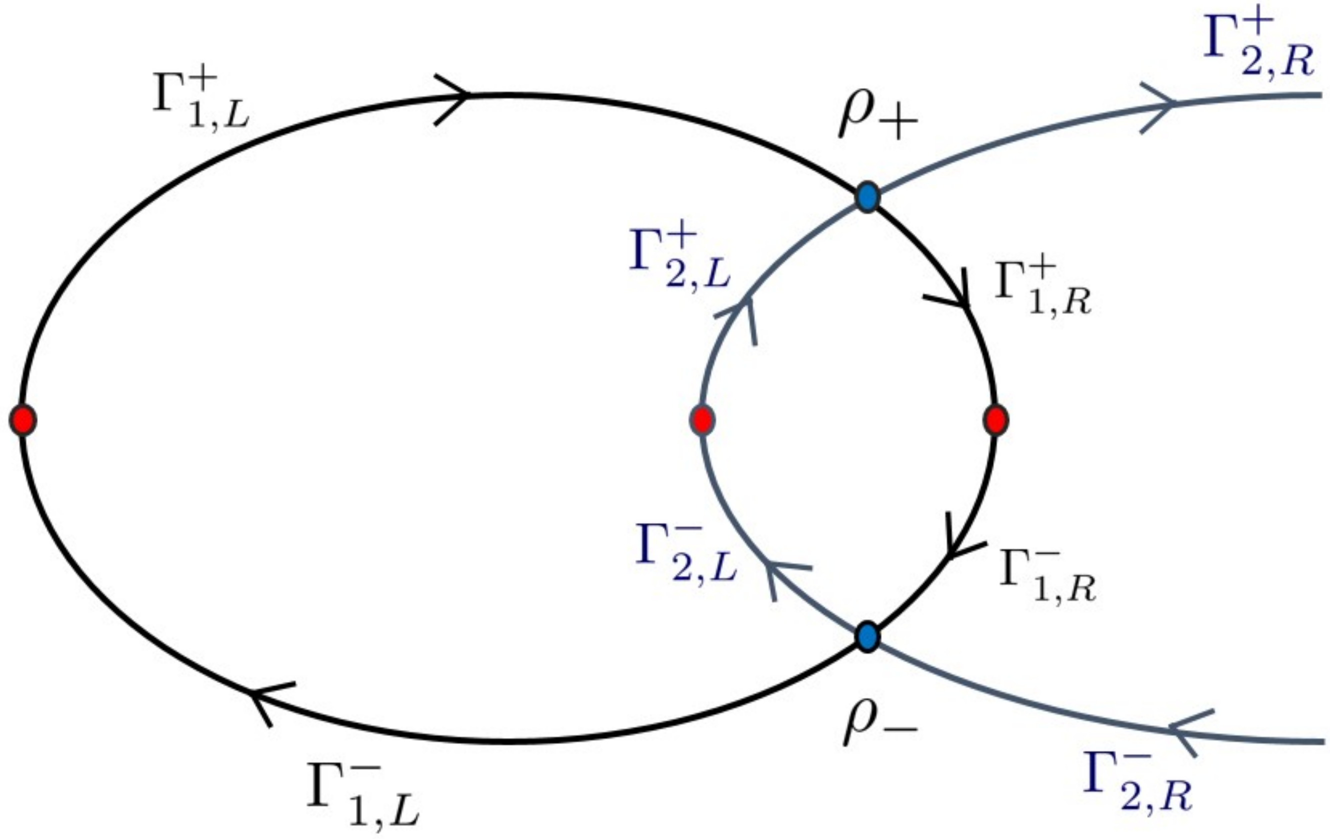
Definition We denote $u \sim 0$ at (x_0, ξ_0) if $\exists \delta > 0$ s.t. as $h \to 0$,

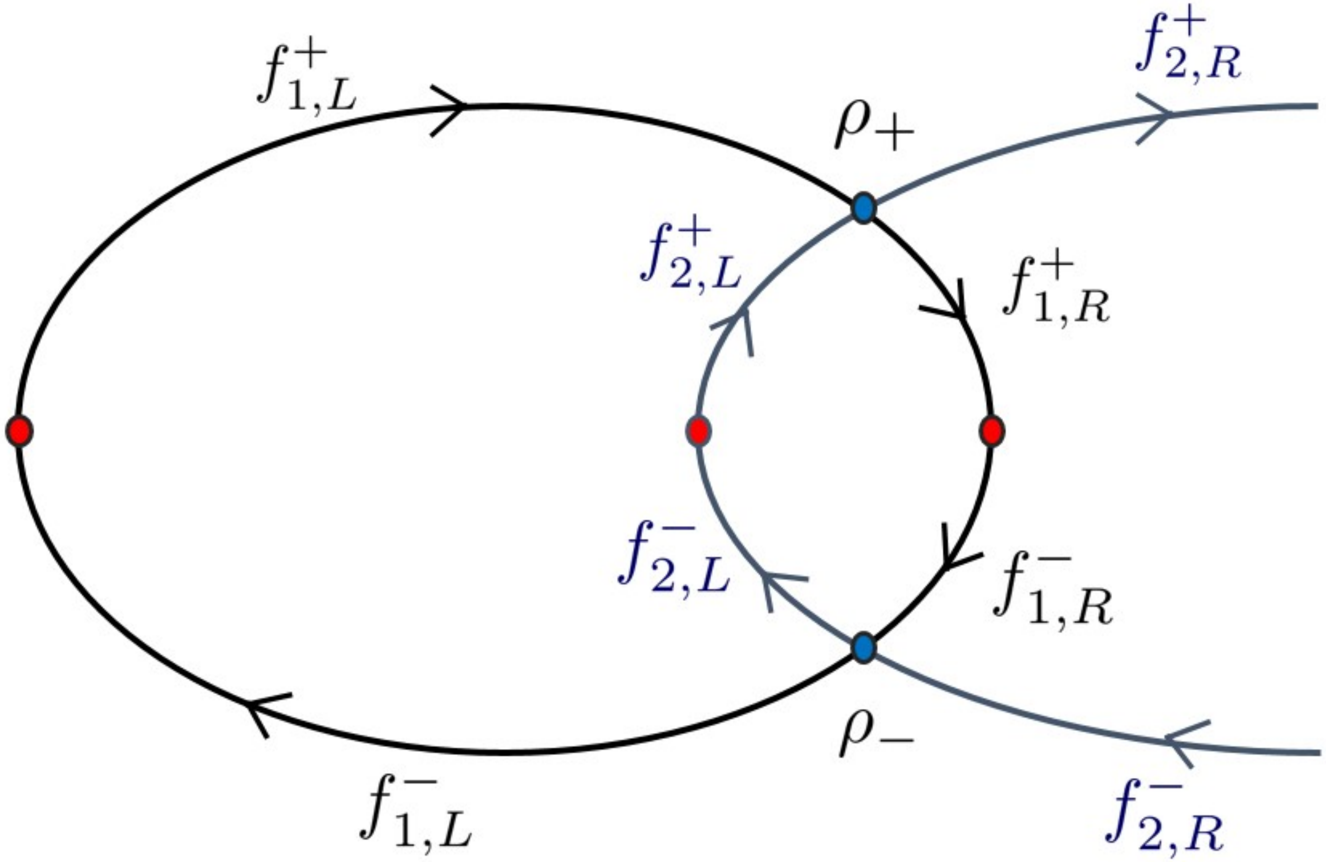
$$(Tu)(x, \xi; h) := \int e^{i(x-y)\xi/h-(x-y)^2/(2h)} u(y) dy = \mathcal{O}(e^{-\delta/h}),$$

uniformly in a neighborhood of (x_0, ξ_0).

We say u is a microlocal solution of $(P - E)u = 0$ in $\Omega \subset \mathbb{R}_x \times \mathbb{R}_\xi$, if $(P - E)u \sim 0$ in Ω.

- If u is a solution to $(P - E)u = 0$, u is microlocally supported on the characteristic set, $\Gamma_j = \{(x, \xi); |\xi|^2 + V_j(x) = E\}, \quad \Gamma = \Gamma_1 \cup \Gamma_2$.
- On Γ, WKB solutions $f^{\pm}_{j,S}$ are microlocally defined on each of 8 curves $\Gamma^{\pm}_{j,S}$ $(j = 1, 2, S = L, R)$, except at
  - 3 turning points $(a(E), 0)$, $(b(E), 0)$ and $(c(E), 0)$
  - 2 crossing points $\rho_+ = (0, \sqrt{E})$ and $\rho_- = (0, -\sqrt{E})$.


$\Gamma_{1,L}^+$
ρ_+
$\Gamma_{2,R}^+$
$\Gamma_{2,L}^+$
$\Gamma_{1,R}^+$
$\Gamma_{2,L}^-$
$\Gamma_{1,R}^-$
ρ_-
$\Gamma_{1,L}^-$
$\Gamma_{2,R}^-$



$f^+_{1,L}$
ρ_+
$f^+_{2,R}$
$f^+_{2,L}$
$f^+_{1,R}$
$f^-_{2,L}$
$f^-_{1,R}$
ρ_-
$f^-_{1,L}$
$f^-_{2,R}$

2.2.1 Microlocal WKB solutions (near $\rho_-(E)$)

On each $\Gamma^{\pm}_{j,S}(E)$ $(j = 1, 2, S = R, L)$, the space of microlocal solutions is one dimensional, and a basis is given by

$$f^{\pm}_{1,S} \sim \begin{pmatrix} a^{\pm}_1 \\ ha^{\pm}_2 \end{pmatrix} e^{\pm i\nu_1(x)/h} \text{ on } \Gamma^{\pm}_{1,S}(E), \quad f^{\pm}_{2,S} \sim \begin{pmatrix} hb^{\pm}_1 \\ b^{\pm}_2 \end{pmatrix} e^{\pm i\nu_2(x)/h} \text{ on } \Gamma^{\pm}_{2,S}(E),$$

Here, for $j = 1, 2$, the phase function is $\nu_j(x) := \int_0^x \sqrt{E - V_j(t)}\, dt$,

$$a^{\pm}_j(x; h) \sim \sum_{k \geq 0} h^k a^{-}_{j,k}(x), \qquad b^{\pm}_j(x; h) \sim \sum_{k \geq 0} h^k b^{-}_{j,k}(x)$$

In particular,

$$a_{1,0} = \frac{1}{(E - V_1)^{\frac{1}{4}}} \;\; ; \;\; a_{2,0} = \frac{r_0 + ir_1\sqrt{E - V_1}}{(V_1 - V_2)(E - V_1)^{\frac{1}{4}}}.$$

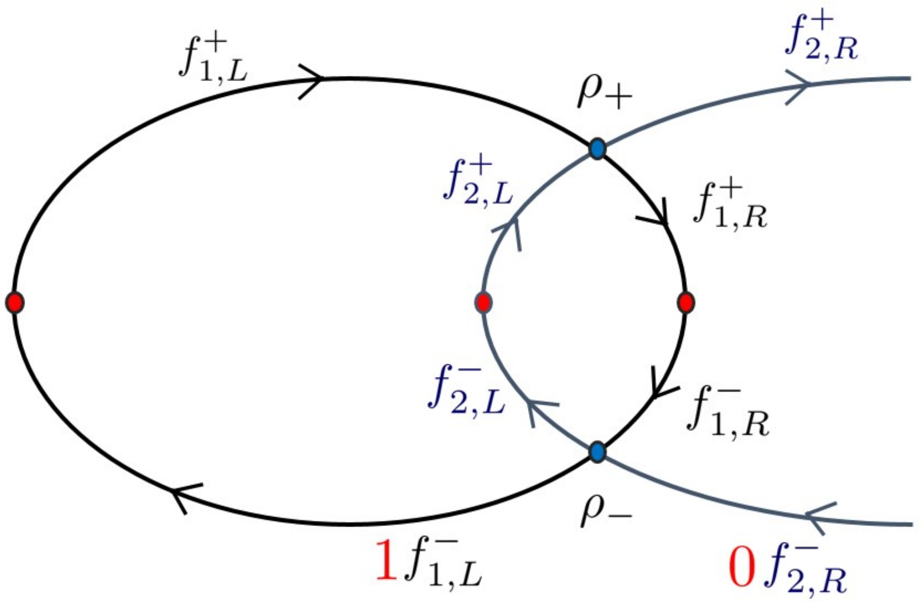
$$b_{2,0} = \frac{1}{(E - V_2)^{\frac{1}{4}}} \;\; ; \;\; b_{1,0} = \frac{r_0 - ir_1\sqrt{E - V_2}}{(V_2 - V_1)(E - V_2)^{\frac{1}{4}}}.$$

These microlocal solutions are not defined at turning points ($V_j = E$) and crossing points ($V_1 = V_2$).

2.2.2 Microlocal connection

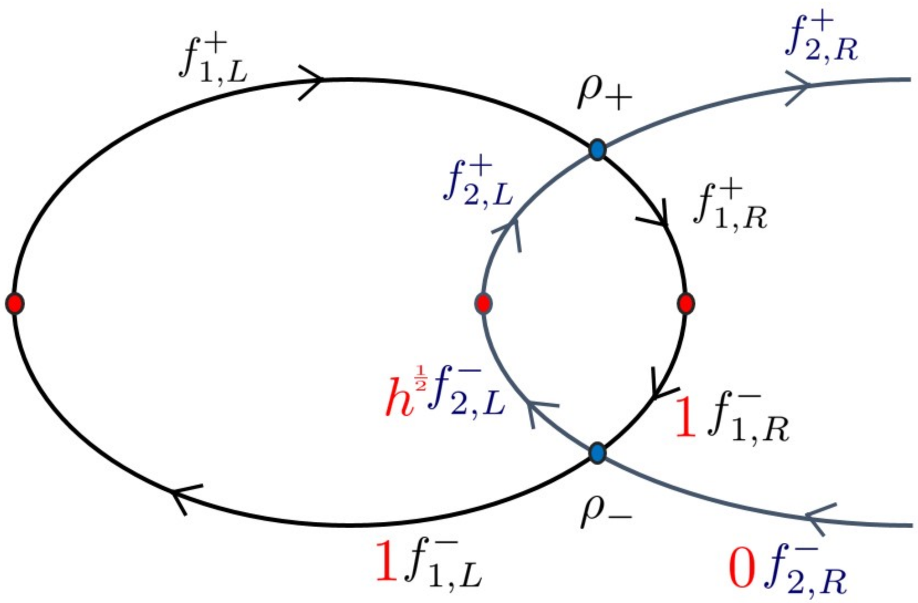
Starting from $\boldsymbol{u} \sim \begin{cases} \boldsymbol{f^-_{1,L}} & \text{on } \boldsymbol{\Gamma^-_{1,L}}, \\ \boldsymbol{0} & \text{on } \boldsymbol{\Gamma^-_{2,R}}, \end{cases}$ we connect the microlocal WKB solutions along the characteristic sets.

Note that, if $\boldsymbol{u}$ is a resonant state, $\boldsymbol{u \sim 0}$ on the incoming trajectory $\boldsymbol{\Gamma^-_{2,R}}$.



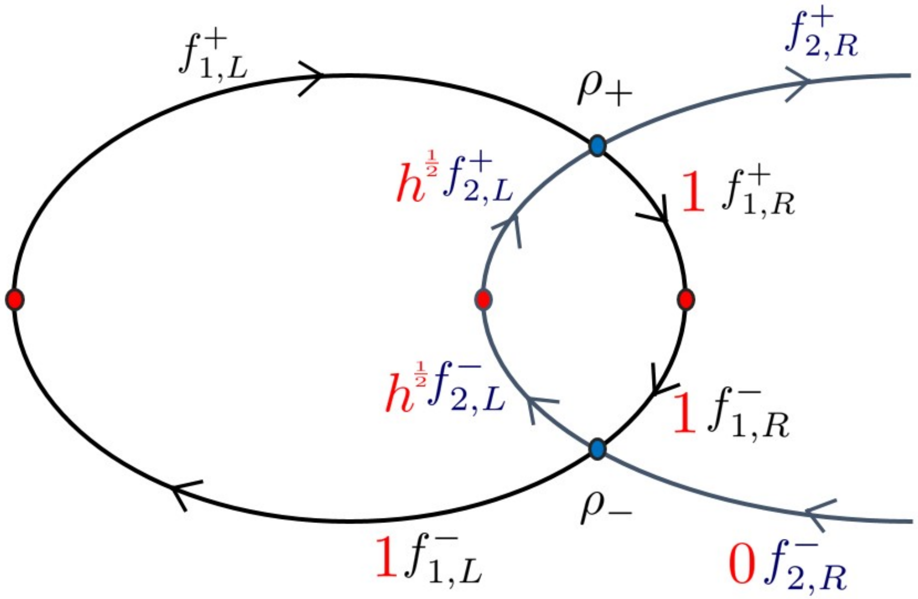
We connect the microlocal WKB solutions over the crossing point $\boldsymbol{\rho_-(E)}$.

Thanks to the ellipticity condition **(A5)**, this problem can be reduced to the connection problem for single-valued equation, whose characteristic sets cross transversally. Such method was given by Helffer-Sjöstrand.



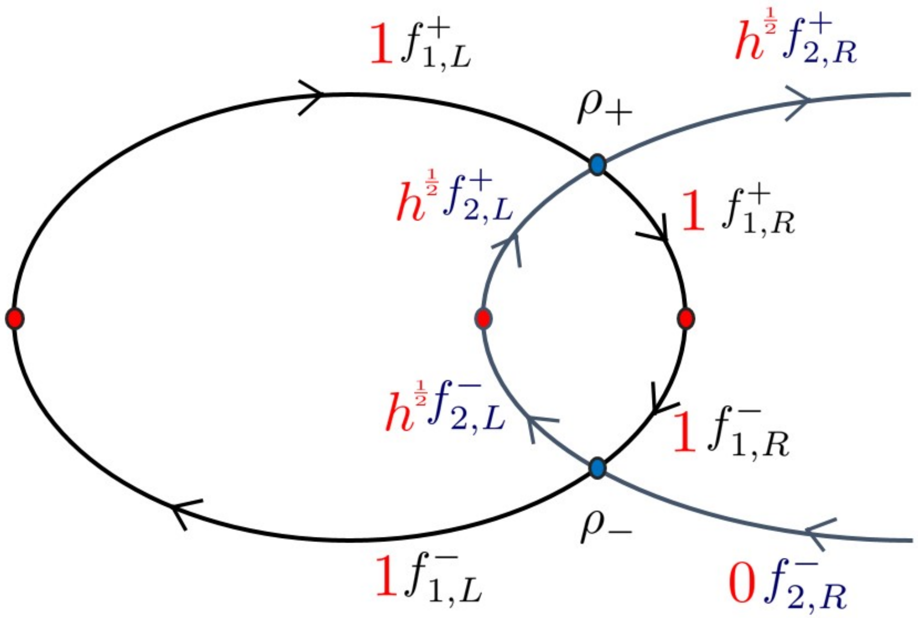
We connect the microlocal WKB solutions over the turning points $(b(E), 0)$ and $(c(E), 0)$.

Maslov theory gives the connection formula through the equation with respect to the momentum variable ξ.



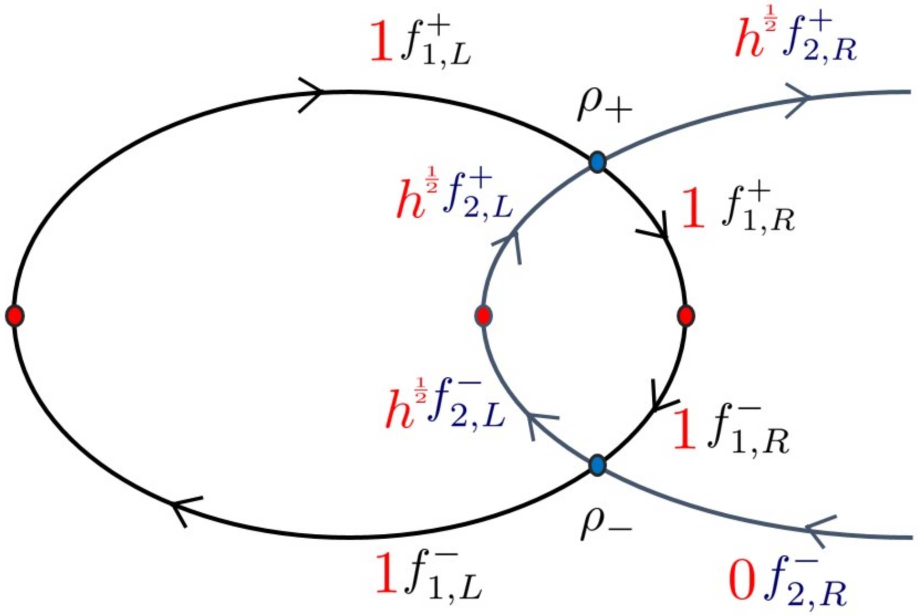
We also connect the microlocal WKB solutions over the crossing point $\boldsymbol{\rho_+(E)}$.

Then we obtain the coefficients $\boldsymbol{t^+_{1,L}}$ and $\boldsymbol{t^+_{2,R}}$ of $\boldsymbol{u} \sim \begin{cases} \boldsymbol{t^+_{1,L} f^+_{1,L}} & \text{on } \boldsymbol{\Gamma^+_{1,L}}, \\ \boldsymbol{t^+_{2,R} f^+_{2,R}} & \text{on } \boldsymbol{\Gamma^+_{2,R}}, \end{cases}$



Continuing u from $\Gamma^+_{1,L}$ to $\Gamma^-_{1,L}$ over the turning point $(a(E), 0)$, we get $u \sim t^-_{1,L} f^-_{1,L}$ on $\Gamma^-_{1,L}$ with $t^-_{1,L} = -e^{2i\mathcal{A}(E)/h} + \mathcal{O}(h)$, and hence more precise Bohr-Sommerfeld quantization rule

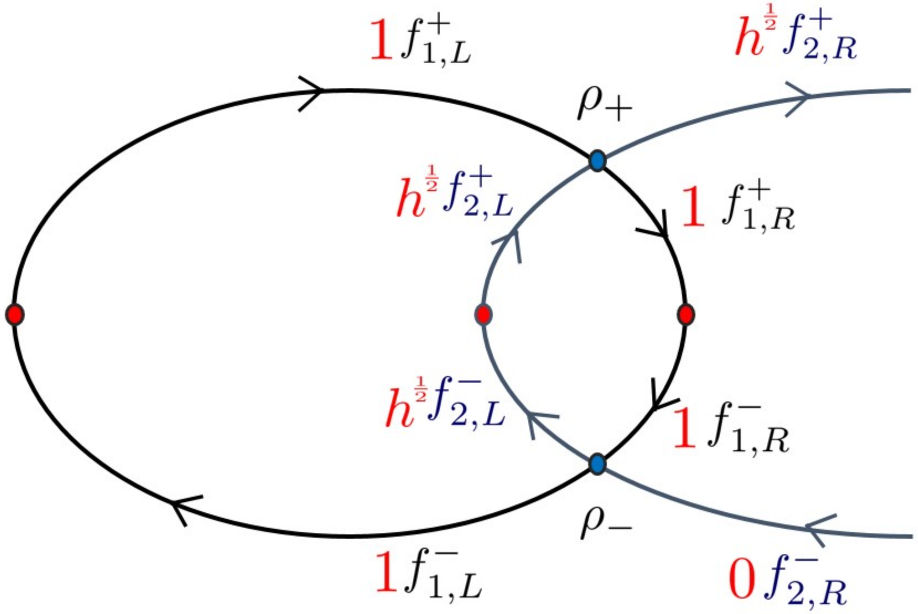
$$-e^{2i\mathcal{A}(E)/h} = 1 + \mathcal{O}(h).$$



Im $\boldsymbol{E}$ is computed from $t^{+}_{2,R}$ via the formula for $x_0 > c(E)$:

$$-\text{Im}\, E = \frac{h^2}{\|u\|^2_{L^2((-\infty,x_0])}} \text{Im}\left(u_1'\overline{u_1} + u_2'\overline{u_2} - r_1 u_2 \overline{u_1}\right)|_{x=x_0}.$$

One sees that, for $E \in \mathcal{D}_h(\delta_0, C_0) \cap \mathbb{R}$, $\|u\|^2_{L^2((-\infty,x_0])} = 2\mathcal{A}'(E)^{1/2} + \mathcal{O}(h^{1/3})$.



Proposition 1 (Via crossing point) Let u be a microlocal solution near $\rho_-(E)$ and suppose

$$u \sim t^-_{j,S} f^-_{j,S} \quad \text{on } \Gamma^-_{j,S}(E),$$

for each $j = 1, 2$, $S = L, R$. Then, for $E \in \mathcal{D}_h(\delta_0, C_0)$, one has

$$\begin{pmatrix} t^-_{1,R} \\ t^-_{2,L} \end{pmatrix} = \begin{pmatrix} \tau^-_{1,1}(E,h) & \tau^-_{1,2}(E,h) \\ \tau^-_{2,1}(E,h) & \tau^-_{2,2}(E,h) \end{pmatrix} \begin{pmatrix} t^-_{1,L} \\ t^-_{2,R} \end{pmatrix},$$

$$\tau^-_{1,1} = 1 + \mathcal{O}(h), \qquad \tau^-_{1,2} = e^{i\frac{\pi}{4}} \tau_- h^{\frac{1}{2}+i\mu h} + \mathcal{O}(h^{\frac{3}{2}}),$$

$$\tau^-_{2,1} = e^{-i\frac{\pi}{4}} \tau_+ h^{\frac{1}{2}-i\mu h} + \mathcal{O}(h^{\frac{3}{2}}), \qquad \tau^-_{2,2} = 1 + \mathcal{O}(h),$$

$$\tau_\pm = \sqrt{\frac{\pi}{\gamma}} \left(r_0(0) E^{-\frac{1}{4}} \pm i r_1(0) E^{\frac{1}{4}} \right).$$

2.2.3 Sketch of the proof of Proposition

(A5) implies $(P-E)v \sim 0 \Leftrightarrow$

$$\begin{cases} Qv_1 \sim 0, \quad Q := W(P_2-E)W^{-1}(P_1-E) - h^2WW^* \\ v_2 \sim Rv_1, \quad R = -h^{-1}W^{-1}(P_1-E) \end{cases}$$

microlocally near $\rho_-(E)$. Since $\rho_-(E)$ is a saddle point of the principal symbol of Q: $(\xi^2+V_1(x)-E)(\xi^2+V_2(x)-E)$, Q is reduced to:

$$UF(Q,h)U^{-1} = \frac{1}{2}(yhD+hDy) \ (= (y\eta)^W),$$

with a Fourier integral operator $Uu = \int_{\mathbb{R}} e^{i\psi(x,y)/h}c(x,y;h)u(y)dy$ and an analytic symbol $F(t,h)$ (Helffer-Sjöstrand).

In our case, we have

$$\psi(x,y) = -\frac{\tau_2}{4\sqrt{E}}x^2 - \frac{\sqrt{E}}{\gamma}y^2 + xy + \sqrt{E}x + \mathcal{O}(|(x,y)|^3),$$

$$F(0,h) = -\frac{i}{2}h + \mu h^2, \quad \mu = -\frac{r_0(0)^2 + r_1(0)^2 E}{2\gamma\sqrt{E}} + \mathcal{O}(h),$$

where $\tau_1 = V_1'(0)$ and $\tau_2 = -V_2'(0)$. The reduced equation for $v = Uu_1$ is

$$\frac{1}{2}(yhD + hDy)v = F(0,h)v$$

and it has a basis of solutions

$$v^{\vdash}(y) = H(y)y^{i\mu h}, \quad v^{\dashv}(y) = H(-y)|y|^{i\mu h}.$$

The problem is thus reduced to the analysis of an integral of the form

$$u_1^{\vdash}(x) := Uv^{\vdash} = \int_0^{\infty} e^{i\psi(x,y)/h}c(x,y;h)y^{i\mu h}dy.$$

Computing the microlocal asymptotics of $u_1^{\vdash}(x)$ and $u_1^{\dashv}(x)$ near $\Gamma_{j,S}$ and comparing with WKB solutions, we obtain

$$v^{\vdash} \sim \begin{cases} \alpha_R^{\vdash}\, h^{i\mu h}\, f_{2,R} & \text{on } \Gamma_{2,R}(E); \\ \alpha_L^{\vdash}\, h^{i\mu h}\, f_{2,L} & \text{on } \Gamma_{2,L}(E); \\ \beta_R^{\vdash}\sqrt{h}\;\; f_{1,R} & \text{on } \Gamma_{1,R}(E); \\ 0 & \text{on } \Gamma_{1,L}(E), \end{cases}$$

where $\alpha_S^{\vdash} \sim \sum_{k\geq 0} h^k \alpha_{S,k}^{\vdash}$ $(S = L, R)$ $\beta_R^{\vdash} \sim \sum_{k\geq 0} h^k \beta_{R,k}^{\vdash}$, and in particular

$$\alpha_{L,0}^{\vdash} = \alpha_{R,0}^{\vdash} = \frac{i\gamma E^{\frac{1}{4}}}{r_0(0) + ir_1(0)\sqrt{E}}; \quad \beta_{R,0}^{\vdash} = \sqrt{\pi\gamma}e^{-i\pi/4}.$$

Similarly, we have

$$v^{\dashv} \sim \begin{cases} \alpha_R^{\dashv} h^{i\mu h} f_{2,R} & \text{on } \Gamma_{2,R}(E) \\ \alpha_L^{\dashv} h^{i\mu h} f_{2,L} & \text{on } \Gamma_{2,L}(E); \\ 0 & \text{on } \Gamma_{1,R}(E); \\ \beta_L^{\dashv} \sqrt{h} \;\; f_{1,L} & \text{on } \Gamma_{1,L}(E), \end{cases}$$

where $\alpha_S^{\dashv} \sim \sum_{k \geq 0} h^k \alpha_{S,k}^{\dashv}$, $\beta_S^{\dashv} \sim \sum_{k \geq 0} h^k \beta_{S,k}^{\dashv}$, for $S = L, R$, and in particular

$$\alpha_{L,0}^{\dashv} = \alpha_{R,0}^{\dashv} = -\frac{i\gamma E^{\frac{1}{4}}}{r_0(0) + ir_1(0)\sqrt{E}}; \quad \beta_{L,0}^{\dashv} = \sqrt{\pi\gamma} e^{i\pi/4}.$$

Proposition 2 (Via turning points) For $S = L, R$, it holds that

$$t^+_{1,S} = \sigma_{1,S} t^-_{1,S}$$

with constants $\sigma_{1,S}$ which behave as $h \to 0$

$$\sigma_{1,L} = -ie^{2iS_{1,L}/h} + \mathcal{O}(h), \qquad \sigma_{1,R} = ie^{-2iS_{1,R}/h} + \mathcal{O}(h),$$

where $S_{1,S}$ are the action integrals defined by

$$S_{1,L}(E) = \int_a^0 \sqrt{E - V_1(t)}dt, \qquad S_{1,R}(E) = \int_0^c \sqrt{E - V_1(t)}dt.$$

Similarly it holds that

$$t^+_{2,L} = \sigma_{2,L} t^-_{2,L}$$

with

$$\sigma_{2,L} = -ie^{2iS_{2,L}/h} + \mathcal{O}(h),$$

where $S_{2,L}$ are the action integrals defined by

$$S_{2,L}(E) = \int_b^0 \sqrt{E - V_2(t)}dt.$$

Thank you for your attention.