Asymptotics for the focusing integrable discrete nonlinear Schrödinger equation

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1. Nonlinear Schrödinger equation and a soliton

focusing NLS

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

soliton

$$u(x,t) = 2\eta e^{2i\xi x - 4i(\xi^2 - \eta^2)t + i(\psi_0 + \pi/2)} \times \operatorname{sech}(2\eta x - 8\xi\eta t - 2\delta_0)$$

carrier wave ($\exp$, oscillatory)$\times$traveling solitary wave (sech)

2. Long-time asymptotics of soliton equations

As $t \to \infty$, the solution is asymptotically
a sum of solitons plus a small perturbation.

NLS: Fokas-Its '96 (IBVP), Kamvissis '95 (IVP)
Toda lattice: Krüger-Teschl '09 (IVP)
KdV: Tanaka '75 (IVP), Grunert-Teschl '09 (IVP)

SOLITON RESOLUTION in recent terminology
(e.g. Terence Tao's "Why are solitons stable?", 2009)
Valid for non-integrable equations as well,
but INTEGRABLE ones are particularly important because

- they are model cases
- phase shift can be written down in the inverse scattering parlance.

3. Integrable Discrete NLS (IDNLS) 1

NLS (**focusing**)

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

Ablowitz-Ladik ('75)
integrable discrete nonlinear Schrödinger equation (**focusing**)

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) + |R_n|^2 (R_{n+1} + R_{n-1}) = 0$$

Both have solitons:
carrier wave ($\exp$, oscillatory)$\times$traveling solitary wave (sech)

4. Integrable discrete NLS (IDNLS) 2

z_1 **eigenvalue**, $|z_1| > 1$,
$C_1(0)$ **norming constant**

a parameter (to be explained laer)

bright soliton

$\mathrm{BS}(n, t; z_1, C_1(0))$
$= (\exp$ carrier wave$) \times (\mathrm{sech}$ traveling wave$)$

If we multiply $C_1(0)$ by another constant
$\Rightarrow$ PHASE SHIFT in $\exp$ and sech.
It happens when solitons collide with one another.

5. Soliton

$R_n(t) = \mathrm{BS}\,(n, t; z_1, C_1(0))$, soliton

$z_1 = \exp(\alpha_1 + i\beta_1)$, $\alpha_1 > 0$: eigenvalue

$C_1(0)$:norming constant (at $t = 0$)

$$\begin{aligned} \mathrm{BS}(n, t; z_1, C_1(0)) &= \text{carrier wave} \times \text{traveling wave} \\ &= \exp\big(-i[2\beta_1(n+1) - 2w_1 t - \arg C_1(0)]\big) \\ &\quad \times \sinh(2\alpha_1)\mathrm{sech}[2\alpha_1(n+1) - 2v_1 t - \boldsymbol{\theta_1}]. \end{aligned}$$

v_1, w_1: written in terms of α_1, β_1.

$\boldsymbol{\theta_1}$.: written in terms of $|C_1(0)|, \alpha_1$.

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6. Soliton collision and phase shift

Popular topic in Integrable systems.

A faster soliton overtakes a slower one.

Velocity and shape are preserved after overtaking.
There may be phase shift, like $f(x - ct) \mapsto f(x - ct + x_0)$.

Multiplication of the norming constant by another constant
$\Rightarrow$phase shift
Different constants as $t \to \infty$ and $t \to -\infty$
$\Rightarrow$phase shift due to collision

Studying phase shift is reduced to studying these constants.

TWO GOALS

- Soliton resolution.
  Is the solution is asymptotically a sum of 1-solitons?
- Calculation of phase shift (as $t \to \infty$)

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7. IDNLS and its Lax pair

$$i\frac{d}{dt}R_n+(R_{n+1}-2R_n+R_{n-1})+|R_n|^2(R_{n+1}+R_{n-1})=0 \quad \text{(IDNLS)}$$

. . . .

n-part : $X_{n+1} = \begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n$

t-part : $\frac{d}{dt}X_n = \left[\text{a complicated matrix} \right] X_n$

(IDNLS) is the compatibility condition.
$z \in \mathbb{C} \setminus \{0\}$ is called the spectral parameter,
Eigenvalues are its special values.

8. Eigenfunctions of the n-part

If $R_n \to 0$ (rapidly) as $n \to \pm\infty$, then approximately

$$X_{n+1} \approx \begin{bmatrix} z & \boxed{0} \\ \boxed{0} & z^{-1} \end{bmatrix} X_n.$$

'solutions' $\begin{bmatrix} z^n \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ z^{-n} \end{bmatrix}$

There exist eigenfunctions
$\phi_n(z), \psi_n(z)$ in $|z| \geq 1$ and $\psi_n^*(z)$ in $|z| \leq 1$
which behave like $z^{\pm n}$ as $n \to \pm\infty$.

3 solutions in the 2-dimensional solution space.
(There's another, but we omit it.)

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9. Eigenvalues and the reflection coefficient

On $|z| = 1$, $\exists a(z)$, $b(z) = b(z,t)$ such that

$$\phi_n(z) = b(z)\psi_n(z) + a(z)\psi_n^*(z),$$

If $a(z_j) = 0$, then $a(-z_j) = 0$.

$\{\pm z_j, \pm \bar{z}_j^{-1}\}$ is called a quartet of eigenvalues.

It corresponds to a soliton.

On $|z| = 1$, the reflection coefficient $r(z)$ is

$$r(z) := \frac{b(z)}{a(z)}$$

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Recall: $\psi_n \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \psi_n^* \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $n \to \infty$.

r is characterized by

$$r\psi_n + \psi_n^* \sim \text{const.} \begin{bmatrix} z^n \\ 0 \end{bmatrix} \quad (n \to -\infty).$$

time evolution

$r(z,t) = r(z) \exp\left(it(z - z^{-1})^2\right)$, where $r(z) = r(z, 0)$.

11. Scattering data

Assume $a(z_j) = 0$ (order 1). $\pm z_j$ is an eigenvalue.
$\phi_n(z_j) = \exists b_j \psi_n(z_j)$.

The norming constant is defined by $C_j := \dfrac{b_j}{\frac{d}{dz}a(z_j)}$

Scattering Data

$$\{(\pm z_j, \pm \bar{z}_j^{-1}, C_j)\}_{j=1}^{J}, \quad r(z)$$

The potential R_n is said to be reflectionless if $r(z) = 0$.

Inverse Scattering Transform

The potential R_n is reconstructed from the scattering data.
Done by using a Riemann-Hilbert problem with poles.

12. Riemann-Hilbert Problem (RHP)

Boundary value problem on the complex plane

Γ: curve (the left-hand side is the $+$ side).

$m(z)$: unknown matrix, components are holomorphic in $\mathbb{C} \setminus \Gamma$

Example: 1. $\Gamma = \mathbb{R}$, $m(z)$ is holomorphic in $\pm \mathrm{Im}\, z > 0$.

2. $\Gamma = \{|z| = 1\}$, $m(z)$ is holomorphic in $|z| \neq 1$.

m_+, m_-: boundary values on Γ from $\pm$ sides

RHP: $m_+ = m_- v$ on Γ (v : **given, JUMP MATRIX**)

No jump if $v = I$. m is analytically continued.

Normalization: $m(z) \to I$ as $z \to \infty$.

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13. RHPs are like integrals

RHP: $m_+ = m_- v$ on Γ

Deift-Zhou's nonlinear steepest descent is based on

- **Contour deformation**, introduction of a new unknown and a new jump matrices

  original RHP $\Leftrightarrow$ new RHP.

- **Continuity** $v \mapsto m$ is continuous.
  (justification of **perturbation analysis**)

- **Removing a part of the contour**

  1. If $v = I$ (no jump) on $\hat{\Gamma} \subset \Gamma$,

     m[original] $= m$[with $\hat{\Gamma}$ deleted]

  2. If $v \approx I$ on $\hat{\Gamma}$, m[original] $\approx m$[with $\hat{\Gamma}$ deleted]

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14. RHP with poles

$m(z)$: unknown matrix, components are meromorphic in $\mathbb{C} \setminus \Gamma$.
RHP: $m_+ = m_- v$ on Γ
$m(z)$ has poles. Residue conditions imposed.

Inverse Scattering

- The jump matrix written in terms of **the reflection coefficient**.
- The poles of $m(z)$ are the **eigenvalues**.
- Residue conditions written in terms of the **norming constants**.

The potential can be reconstructed from the solution $m(z)$ of the RHP.

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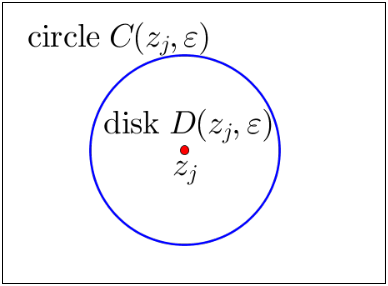
15. ‘blowup’ of poles

$$\operatorname{Res}(m(z); z_j) = \lim_{z \to z_j} m(z) \begin{bmatrix} 0 & 0 \\ z_j^{-2n} C_j(t) & 0 \end{bmatrix}.$$

Replace a pole with a circle.
Then one can use the RHPs-are-like-integrals technique.



In the disk $D(z_j, \varepsilon)$,

subtract the singular part of m.

New unknown $\hat{m}$.

holomorphic at z_j.

jump along $C(z_j, \varepsilon)$ instead.

If the jump matrix is close to the identity, it can be ignored.
Decrease $\sharp$ of poles $\Rightarrow$ reduction to the 1-soliton case

16. IVP of IDNLS

Time evolution of the scattering data

Eigenvalues are independent of time.

$$C_j(t) = C_j(0) \exp\left(it(z_j - z_j^{-1})^2\right),$$

$r(z,t) = r(z) \exp\left(it(z - z^{-1})^2\right)$ on $|z| = 1$,

where $r(z) := r(z, 0)$

Initial value problem

$R_n(0)$ determines the scattering data at $t = 0$.
The scattering data in $t > 0$ are determined.
Formulate RHP with poles (involving the scattering data).
The potential $R_n(t)$ $(t > 0)$ is reconstructed from the solution of the RHP.

17. Reflectionless Case

If $\boxed{r(z) = r(z,0) = 0}$, $R_n(t) =$ multi-soliton.
It approaches a sum of 1-solitons as $t \to \infty$.

phase shift (formal proof in Ablowitz-Prinari-Trubatch '04)

Each term is of the form $\mathrm{BS}(n, t, z_j, \underline{p_j T(z_j)^{-2}} C_j(0))$

phase shift

Phase shift is determined by the eigenvalues:

$$p_j := \prod_{k>j} z_k^2 \bar{z}_k^{-2}, \qquad T(z_j) := \prod_{k>j} \frac{z_k^2 (z_j^2 - \bar{z}_k^{-2})}{z_j^2 - z_k^{-2}}$$

The j-th soliton is faster than the $(j-1)$-th.

Q1 Is soliton resolution valid even if $r(z) \neq 0$?

Q2 How does the reflection affect the phase shift?

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Q2 **How does the reflection affect the phase shift?**

18. Main results (sketched)

What happens as $t \to \infty$ if there *is* reflection ($r(z) \not\equiv 0$)?
Some generic assumptions (finite number of eigenvalues, ...)

SOLITON RESOLUTION
A sum of 1-solitons plus a small perturbation

A new PHASE SHIFT formula involving the REFLECTION COEFFICIENT $r(z)$.

$|n|/t < 2$ (the 'timelike' region)

There is a new factor written in terms of $r(z)$:
$\mathrm{BS}(n, t, z_j, \text{New} \cdot p_j T(z_j)^{-2} C_j(0))$

$|n|/t \geq 2$

Leading term is the same as in the reflectionless case.

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19. Asymptotic Behavior: $r(z) \neq 0$

♣ **tw** is the velocity of the soliton (traveling wave).

$\boxed{|\mathrm{tw}(z_j)| < 2}$ *Timelike Region: New Phase Shift Formula*

$$R_n(t) = \mathrm{BS}\left(n, t; z_j, \boldsymbol{\delta(0)\delta(z_j)^{-2}} p_j T(z_j)^{-2} C_j(0)\right) + O(t^{-1/2}).$$

$\boldsymbol{\delta(z)}$ determined by $r(z)$.
p_j, $T(z_j)$ determined by z_k's $(k \geq j)$.
z_k's correspond to the j-th and faster solitons.

$\boxed{|\mathrm{tw}(z_j)| = 2}$ Leading term remains the same

$$R_n(t) = \mathrm{BS}\left(n, t; z_j, p_j T(z_j)^{-2} C_j(0)\right) + O(t^{-1/3}).$$

$\boxed{|\mathrm{tw}(z_j)| > 2}$ Leading term remains the same

As $|n| \to \infty$,

$$R_n(t) = \mathrm{BS}\left(n, t; z_j, p_j T(z_j)^{-2} C_j(0)\right) + O(n^{-k}), \quad \forall k.$$

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20. function $\delta(z)$ in the timelike region

In $|n| < 2t$ (the timelike region),

$$A := \frac{1}{2}\left(\sqrt{2+n/t} - i\sqrt{2-n/t}\right).$$

$$S_1 := e^{-\pi i/4}A,\ S_2 := e^{-\pi i/4}\bar{A},\ S_3 := -S_1,\ S_4 := -S_2.$$

all on $|z| = 1$.

Saddle points of the phase function to be explained later.

$$\delta(z) := \exp\left(\frac{-1}{2\pi i}\left[\int_{S_1}^{S_2} + \int_{S_3}^{S_4}\right](\tau - z)^{-1}\log(1+|r(\tau)|^2)\,d\tau\right)$$

$\delta(z)$ is determined by the **reflection coefficient**.
$\delta(z) \equiv 1$ in the reflectionless case.

21. RHP with poles and phase function

$m_+(z) = m_-(z)v(z)$ on $|z| = 1$

$$v(z) = \begin{bmatrix} 1 + |r(z)|^2 & e^{-2\varphi}\bar{r}(z) \\ e^{2\varphi} r(z) & 1 \end{bmatrix}$$ JUMP MATRIX

$$\varphi = \frac{1}{2} it(z - z^{-1})^2 - n \log z$$ *PHASE FUNCTION!*

"Nonlinear Fourier-Laplace analysis"

Residue conditions at the poles of $m(z)$ written in terms of the norming constants.

Potential reconstruction $\quad R_n(t) = -\frac{d}{dz} m(z)_{21}\Big|_{z=0}$

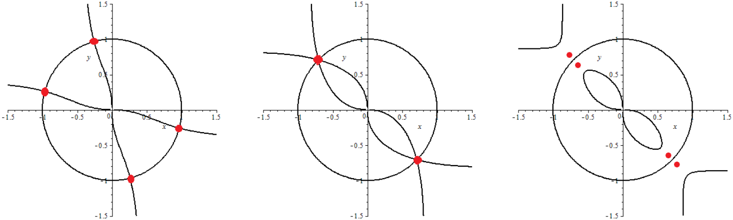
IVP solved. Asymptotic expansion calculated.

22. Different behaviors in different regions. Why?

$\varphi = \frac{1}{2} it(z - z^{-1})^2 - n \log z$ phase function

$\{\mathrm{Re}\, \varphi(z) = 0\} \supset \{|z| = 1\}$. Red dots are stationary points.

$\boxed{|n| < 2t}$ $\boxed{|n| = 2t}$ $\boxed{|n| > 2t}$



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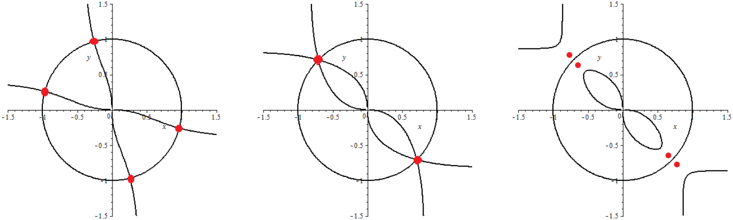
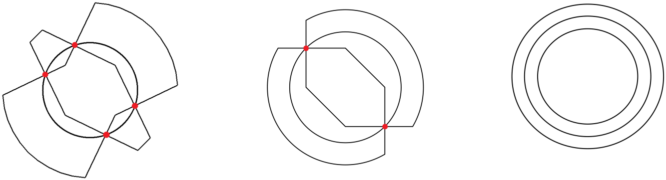


Steepest descent paths.



Thank you very much.

- Long-time asymptotics for the integrable discrete nonlinear Schrödinger equation: the focusing case, to appear in Funkcialaj Ekvacioj, arXiv:1512.01760 [math-ph]
- (related work) Riemann-Hilbert factorization of matrices invariant under inversion in a circle, to appear in Proc. AMS, Volume 147, Number 5, May 2019, arXiv:1805.12366 [math-ph]