Calabi-Yau structure and Bargmann type transformation on the Cayley projective plane

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Introduction

Geometric structure on a manifold defines a differential or integral operator. Subject today is to present a non-trivial example of an operator defined by polarizations on a particular symplectic manifold. The method for this purpose is pairing of polarizations. The operator I call a **Bargmann type transformation**.

Let X be a symplectic manifold with a symplectic form ω^X whose cohomology class $[\omega^X] \in H^2(X, \mathbb{Z})$:
Roughly, $\mathcal{F} \subset T(X) \otimes \mathbb{C}$ is a polarization, if $\omega^X{}_{|\mathcal{F}\times\mathcal{F}} = 0$ and $2\dim \mathcal{F} = \dim X$ (with integrability condition).
Typical polarizations are real and Kähler polarization.

- real polarization = Lagrangian foliation
- Kähler polarization = X is a Kähler manifold.

We have many Kähler manifolds and cotangent bundles always have the real polarization. We consider a case which has both of such structures, then the constructed operator maps from a space of "polarized sections" to another space of polarized sections by another "polarization".

So, here the symplectic manifold is the cotangent bundle of "*Cayley projective plane*" (precisely to say, zero section must be removed). The Cayley projective plane is a manifold in the class, so called, *rank one compact symmetric space*:

$$\boldsymbol{ROCSS} = \{S^n, P^n\mathbb{R}, P^n\mathbb{C}, P^n\mathbb{H}, P^2\mathbb{O}\}.$$

A reason I tried to treat this space is that it remains only the case in this class not to be given explicitly the Bargmann type transformation (within my knowledge). So I was curious how it looks like and the restricted daily life forced by the new corona was a good occasion.

The main points are

(1) to find an explicit form of a nowhere vanishing global holomorphic $\mathbf{16}$*-form of the canonical line bundle of* $T_0^*(P^2\mathbb{O})$*, which is realized as a quadric submanifold* $\subset \mathbb{C}^{27}\backslash\{0\}$*,*

(2) the charactrization of polynomials, which I call "Cayley harmonic polynomials", similar to harmonic polynomials.

As an application, the transformation gives a quantization of the geodesic flow in terms of one parameter group of elliptic Fourier integral operators whose canonical relations are defined by the graphs of the geodesic flow action at each time.

This is based on a joint work with Kurando Baba (former colleague at TUS): arXiv 2101.07505: Calabi-Yau structure and Bargmann type transformation on the Cayley projective plane

- Classical Bargmann transformation

Fock space : $\mathcal{F} = \Big\{ f$: holomorphic on $\mathbb{C}^n \Big|$

$$\|f\|^2 = \int_{\mathbb{C}^n} |f(z)|^2 \cdot e^{-\pi|z|^2} dx \wedge dy < \infty \Big\},$$

$$T : \mathcal{F} \longrightarrow L_2(\mathbb{R}^n),$$

$$T(f)(x) = \int_{\mathbb{R}^n} f(x + \sqrt{-1}y) \cdot e^{-\pi(|x|^2+|y|^2)/2 - \sqrt{-1}\pi<x,y>} dy.$$

$\mathcal{F} \ni f$ can be seen as a good classical observable, especially $T(f)$ is an eigenstate of the harmonic oscillator, if f is a homogeneous polynomial on $\mathbb{C}^n$.

To consider this transformation we need several quantities defined by geometric structures on the configuration space $\mathbb{R}^n$ and its phase space $T^*(\mathbb{R}^n)$:

(1) a Kähler structure on $T^*(\mathbb{R}^n)$, which is given by identifying as $T^*(\mathbb{R}^n) \cong \mathbb{C}^n$,

(2) the measure $e^{-\pi|z|^2} dx \wedge dy$ to make the space of good classical observables into a Hilbert space,

(3) the kernel function $e^{-\pi(|x|^2+|y|^2)/2-\sqrt{-1}\pi<x,y>}$ where or how we can choose?

(4) the meaning of the integral

The original transformation was for the Euclidean space and I do not know whether it is possible to construct for any closed manifold.

Possible cases I know are the manifolds in *ROCSS* except $P^2\mathbb{O}$, that is,

(1) sphere S^n by J.H. Rawnsley; Trans. Amer. Math. Soc. **250** (1979), 167–180.
(2) complex projective spaces $P^n\mathbb{C}$, by K.F. and S. Yoshizawa; Japanese J. Math. **21**(1995), 355–392.
(3) quaternion projective space $P^n\mathbb{H}$, by K. F.; Annals of Global Analysis and Geometry, Vol. 22, No. 1, 1–27(2002).

We can construct Bargmann type transformation for complex and quaternion projective spaces using Hopf fiberations, however for Calyley projective plane we have no such principal bundle structure with the total space being a sphere.

So we go back to the sphere case how are compact space cases.

- Sphere case

$$S^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \|x\| = 1 \right\},$$

$$T^*(S^n) \cong \left\{ (x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid x \in S^n, < x, y >= 0, \right\},$$

- $\tau_S : T^*(S^n) \ni (x, y) \longmapsto z = \|y\| x + \sqrt{-1} y \in \mathbb{C}^{n+1},$

$$\tau_S \left(T^*_0(S^n) \right) = \left\{ z \in \mathbb{C}^{n+1} \;\middle|\; z^2 := \sum z_i{}^2 = 0, z \neq 0 \right\}$$

this is a quadric hyper-surface in $\mathbb{C}^{n+1} := Q_2$, then

- $\tau_S{}^*(\sqrt{-1} \, \bar{\partial} \partial \, \|z\|) = \omega^S$: symplectic form on $T^*(S^n)$
- the canonical line bundle : $\bigwedge^{max} T^*(Q_2) \otimes \mathbb{C}^{(1,0)}$

$:= \bigwedge^{max} T^{*'}(Q_2)$ is holomorphically trivial.

These in mind I will explain our case.

§1 Octanion $\mathbb{O}$ and Cayley projective plane $P^2\mathbb{O}$

§2 Kähler structure and Calabi-Yau structure on the space = punctured cotangent bundle $T_0^*(P^2\mathbb{O})$

§3 Fock(like) spaces

§4 Bargmann type transformation $\mathfrak{B}$

§5 Outline of a proof of the main theorem

Octanion $\mathbb{O}$ and Cayley projective plane $P^2\mathbb{O}$

- $\mathbb{O} = \mathbb{H} + \mathbb{H}e_4$: **octanion number field** $\cong \mathbb{R}^8$
- $\mathbb{O} \ni h = \sum_{i=0}^{7} h_i \mathrm{e}_i, h_i \in \mathbb{R}, \ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O} \ni h = \sum_{i=0}^{7} h_i \mathrm{e}_i, h_i \in \mathbb{C},$
- $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O} \cong \mathbb{C}(2) \oplus \mathbb{C}(2),$ since $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{C}(2),$
- $\overline{h} = \sum_{i=0}^{7} \overline{h_i} \mathrm{e}_i, \qquad \theta(h) = h_0 \mathrm{e}_0 - \sum_{i=1}^{7} h_i \mathrm{e}_i,$
- $\sum_i h_i^2 = h\theta(h) = \theta(h)h$

- **A Jordan algebra $\mathcal{J}(3)$**

$$\mathcal{J}(3) = \left\{ \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \,\middle|\, \xi_i \in \mathbb{R},\ x, y, z \in \mathbb{O} \right\} \cong \mathbb{R}^{27}$$

Jordan product : $X \circ Y := \dfrac{XY + YX}{2}$

inner product $< \cdot, \cdot >^{\mathcal{J}(3)}$ in $\mathcal{J}(3)$:

$$< X, X' >^{\mathcal{J}(3)} = \mathrm{tr}(X \circ X')$$

$$= \sum \xi_i \eta_i + 2(< x, x' > + < y, y' > + (z, z' >),$$

the exceptional group F_4 is defined as the automorphism group of $\mathcal{J}(3)$, $\quad F_4 :$

$$= \{\alpha \in SO(27) \mid \alpha(X \circ Y) = \alpha(X) \circ \alpha(Y), \alpha(Id) = Id\}$$

Cayley projective plane:= $P^2\mathbb{O}$

$$P^2\mathbb{O} = \left\{X \in \mathcal{J}(3) \mid X^2 = X, \mathrm{tr}(X) = \sum \xi_i = 1\right\},$$

$P^2\mathbb{O} \cong F_4/Spin(9)$ as the compact symmetric space, where $Spin(9) \cong$ the starionary subgroup at $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\dim P^2\mathbb{O} = 16.$

$$T(P^2\mathbb{O}) = \{(X, Y) \in \mathcal{J}(3) \times \mathcal{J}(3) \mid X \circ Y = Y/2\}$$

$$\tau_{\mathbb{O}} : T(P^2\mathbb{O}) \cong T^*(P^2\mathbb{O}) \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathcal{J}(3),$$

$$\tau_{\mathbb{O}}(X, Y) = \|Y\|^2 X - Y^2 + \sqrt{-1} \otimes \frac{\|Y\|}{\sqrt{2}} Y,$$ then

Proposition

(1) The map $\tau_{\mathbb{O}}$ is an isomorphism on $T^*_0 P^2\mathbb{O}$:

$$T^*_0(P^2\mathbb{O}) \overset{\tau_{\mathbb{O}}}{\cong} \mathbb{X}_{\mathbb{O}} = \{A \in \mathbb{C} \otimes \mathcal{J}(3) \mid A^2 = 0, A \neq 0\}.$$

(2) $\tau_{\mathbb{O}}{}^*(\sqrt{-2}\,\overline{\partial}\,\partial\,\|A\|^{1/2}) = \omega^{P^2\mathbb{O}}$: the natural symplectic form

Only when we consider the map $\tau_{\mathbb{O}}$ on $T^*_0 P^2\mathbb{O}$ it can be diffeomorphic. This is common among projective spaces, different form the Euclidean case.

Calabi-Yau structure on $T^*_0(P^2\mathbb{O})$

This is used for defining Fock(-like) spaces on $\tau_{\mathbb{O}}(T^*_0 P^2\mathbb{O}) = \mathbb{X}_{\mathbb{O}}$.

Theorem

(1) The space $\mathbb{X}_{\mathbb{O}}$ is a complex 16-dimensional smooth affine algebraic manifold defined as the intersection of null sets of 27 quadric equations (+ one linear function $\mathrm{tr}(A) = \xi_1 + \xi_2 + \xi_2 = 0$, although this is implicitly included in 27 quadric equations).

(2) Let $\pi : \mathbb{X}_{\mathbb{O}} \to \mathbb{C}^* \backslash \mathbb{X}_{\mathbb{O}} := \overline{\mathbb{X}_{\mathbb{O}}}$:non-singular submanifold $\subset P^{26}\mathbb{C}$. Let $\mathcal{V} = \mathrm{Ker}\,(d\pi) \subset T(\mathbb{X}_{\mathbb{O}})$. This can be seen as a trivial complex line bundle and we have an isomorphism:

$$\bigwedge^{max} T^{*(1,0)}(\mathbb{X}_{\mathbb{O}}) \cong \bigotimes^{5} \pi^* \left(\mathcal{L}_{|\overline{\mathbb{X}_{\mathbb{O}}}} \right) \otimes \mathcal{V}^*,$$

where $\mathcal{L}$ is the tautological line bundle on $P^{26}\mathbb{C}$.

Its pull back by $\mathbb{C}^{27}\backslash\{0\} \to P^{26}\mathbb{C}$ is trivial line bundle.
The number **5** is obtained by determining the Jacobian explicitly.

Among the matrix form of equation $A^2 = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix}^2 = 0,$

as an example I take one equation

$(\xi_2 + \xi_3)x + \theta(yz) = 0,$ whose 2×2 matrix form is

$$\xi_1 \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} - \begin{pmatrix} w_4 & -w_2 \\ -w_3 & w_1 \end{pmatrix}\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$$

So if we assume $z_1 \neq 0$, then we can solve y_1 and y_3 in terms of some combination of 16 variables including $z_1 \neq 0$. Likewise if we assume $v_1 \neq 0$, then we can solve w_4 and w_3 in terms of some 16 variables including v_1.

These give us an open covering by complex coordinates neighborhood, each has the form $\mathbb{C}^* \times \mathbb{C}^{15}$, and we calculate the Jacobian J explicitly, then it has a form of $J = \frac{g}{g'}$ (in this case $J = \left(\frac{v_1}{z_1}\right)^5$ with the nowhere vanishing holomorphic functions z_1 and v_1) defined on the whole coordinate neighborhood.
The number of such Jacobian is ${}_{24}C_2 = 12 \times 23$, but it is enough to calculate at most 3.

Definition

We put the nowhere vanishing global holomorphic 16-form defined by this way by $\Omega^{\mathbb{X}_{\mathbb{O}}}$.

In fact, I write a part of $\Omega^{\mathbb{X}_{\mathbb{O}}}$:it is expressed on $U_{z_1} = \{A \mid z_1 \neq 0\}$, and there

$$\Omega^{\mathbb{X}_{\mathbb{O}}} = \frac{1}{{z_1}^5} dz_1 \wedge dz_2 \wedge dz_3 \wedge dw_1 \wedge \cdots \wedge dw_4 \wedge dx_3 \wedge dx_4 \wedge \cdots .$$

Theorem (continued)

Then

(1) $\Omega^{\mathbb{X}_{\mathbb{O}}}$ is F_4-invariant and we have the relation

(2) $${\tau_{\mathbb{O}}}^{*}\left(\Omega^{\mathbb{X}_{\mathbb{O}}} \bigwedge \overline{\Omega^{\mathbb{X}_{\mathbb{O}}}}\right) = 2^{26} \cdot \|Y\|^{28} \cdot \frac{1}{16!}(\omega^{P^2\mathbb{O}})^{16}.$$

(the last term is the Liouville volume form)

Let $\mathbf{q} : T^*(P^2\mathbb{O}) \to P^2\mathbb{O}$ be the projection map and let $dv_{P^2\mathbb{O}}$ be the Riemann volume form on $P^2\mathbb{O}$, then one more relation I mention

Proposition

$$\mathbf{q}^*(dv_{P^2\mathbb{O}}) \bigwedge {\tau_{\mathbb{O}}}^*\left(\overline{\Omega^{\mathbb{X}_{\mathbb{O}}}}\right) = 2^6 \cdot ||Y||^6 \cdot \frac{1}{16!}\left(\omega^{P^2\mathbb{O}}\right)^{16}.$$

Fock spaces

- $\mathbb{L}$: the line bundle corresponding to the symplectic form $\omega^{P^2\mathbb{O}}$.

Consider the space (or subspaces) of sections

$$\Gamma_{\mathcal{G}}\Big(\mathbb{L}\otimes \bigwedge^{max} T^{*(1,0)}(\mathbb{X}_{\mathbb{O}})\Big),$$

where $\Gamma_{\mathcal{G}}$ means parallel(or polarized) sections with respect to the "*partial connection*" defined by the Kähler polarization $\mathcal{G}$:

$$\mathcal{G} = T^{(0,1)}(\mathbb{X}_{\mathbb{O}}).$$

Polynomials are seen as sections of this space by the map

$$\sum \mathcal{P}_k \ni p \longmapsto p \cdot \mathbf{t}_0 \otimes \Omega^{\mathbb{X}_{\mathbb{O}}},$$

where $\mathbf{t}_0 \in \Gamma(\mathbb{L})$, parallel w.r.t. the complex polarization:

$$\nabla_X(\mathbf{t}_0) = 0,\ X \in \Gamma(T^{(0,1)}(\mathbb{X}_{\mathbb{O}}).$$

• We introduce inner products in this space (in subspaces).
Naturally the inner product of two sections $p_i \cdot \mathbf{t}_0 \otimes \Omega^{\mathbb{X}_{\mathbb{O}}}$ will be

$$(p_1, p_2)_\epsilon := \left(p_1 \cdot \mathbf{t}_0 \otimes \Omega^{\mathbb{X}_{\mathbb{O}}}, p_2 \cdot \mathbf{t}_0 \otimes \Omega^{\mathbb{X}_{\mathbb{O}}}\right)$$
$$= \int_{\mathbb{X}_{\mathbb{O}}} p_1 \overline{p_2} < \mathbf{t}_0, \mathbf{t}_0 > \cdot \, ||A||^\epsilon \cdot \Omega^{\mathbb{X}_{\mathbb{O}}} \bigwedge \overline{\Omega^{\mathbb{X}_{\mathbb{O}}}}$$
$$= \int_{\mathbb{X}_{\mathbb{O}}} p_1 \overline{p_2} \cdot e^{-\sqrt{2}\pi ||A||^{1/2}} \cdot ||A||^\epsilon \cdot \Omega^{\mathbb{X}_{\mathbb{O}}} \bigwedge \overline{\Omega^{\mathbb{X}_{\mathbb{O}}}},$$

except the additional factor $||A||^\epsilon$.
The factor $e^{-\sqrt{2}\pi ||A||^{1/2}} = e^{2\pi\sqrt{-1}\lambda} =< \mathbf{t}_0, \mathbf{t}_0 >$ is given by solving the equation

$$d\lambda = \tau_{\mathbb{O}}^* \left(\sqrt{-2}\partial \, ||A||^{1/2}\right) - \theta^{P^2\mathbb{O}}$$

and this comes from the difference (= the equation above) of unitary and holomorphic trivialization of the line bundle $\mathbb{L}$.

Definition: Fock space $\mathcal{F}_\epsilon$

We denote by $\mathcal{F}_\epsilon$, the completion of the space

$$\left\{ p \cdot \mathbf{t}_0 \otimes \Omega^{\mathbb{X}_{\mathbb{O}}} \,\middle|\, p \in \mathcal{P}[\mathcal{J}(3)] \right\}$$

with respect to the inner product $(\cdot, \cdot)_\epsilon$.

We regard it is a completion of the polynomial space $\mathcal{C}[\mathcal{J}(3)]$ through the correspondence

$$\mathcal{C}[\mathcal{J}(3)] \ni p \longmapsto p \cdot \mathbf{t}_0 \otimes \Omega^{\mathbb{X}_{\mathbb{O}}}.$$

Bargmann type transformation $\mathfrak{B}$

$$\mathfrak{B} : \sum_{k=0}^{\infty} \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}] \longrightarrow C^{\infty}(P^2\mathbb{O}),$$

$$\mathfrak{B}(h) \cdot dv_{P^2\mathbb{O}}$$

$$= \{\mathbf{q} \circ (\tau_{\mathbb{O}})^{-1}\}_*(h \cdot e^{-\sqrt{2}\pi\|A\|^{1/2}}\pi^*(dv_{P^2\mathbb{O}}) \wedge \Omega^{\mathbb{X}_{\mathbb{O}}}),$$

$$\sum_{k=0}^{\infty} \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$$

is the space of restrictions of polynomials $\mathbb{C}[\mathcal{J}(3)]$ to $\mathbb{X}_{\mathbb{O}}$ and

$$\{\mathbf{q} \circ (\tau_{\mathbb{O}})^{-1}\}_*$$

means the integrals along the fiber $\mathbf{q}^{-1}(\mathbf{q} \circ (\tau_{\mathbb{O}})^{-1}(A))$.

The transformation $\mathfrak{B}$ is originally defined as a sesqui-bilinear form on

$$\Gamma_{\mathcal{L}}\Big(\mathbb{L}\otimes \mathbf{q}^*(\overset{16}{\wedge} T^*(P^2\mathbb{O}))\Big) \times \Gamma_{\mathcal{G}}\Big(\mathbb{L}\otimes \overset{max}{\wedge} T^{*(1,0)}(\mathbb{X}_{\mathbb{O}})\Big),$$

$\mathcal{L}$ is the real polarization on $T^*_0(P^2\mathbb{O})$ and so $C^\infty(P^2\mathbb{O}) \approx \Gamma_{\mathcal{L}}(\mathbb{L}\otimes|\wedge|^{1/2}(\mathbb{X}_{\mathbb{O}}))$.

This sesqui-linear form is understood as

$$C^\infty(P^2\mathbb{O})\times\mathbb{C}[\mathcal{J}(3)] \ni (f,h) \mapsto \int_{P^2\mathbb{O}} f\cdot\overline{\mathfrak{B}(h)}dv_{P^2\mathbb{O}}$$

$$= \int_{\mathbb{X}_{\mathbb{O}}} \mathbf{q}^*(f)(X,Y)\cdot\overline{h(\tau_{\mathbb{O}}(X,Y))}\times$$

$$\times\, e^{-\sqrt{2}\pi||Y||}\cdot \mathbf{q}^*(dv_{P^2\mathbb{O}})\wedge {\tau_{\mathbb{O}}}^*(\overline{\Omega^{\mathbb{X}_{\mathbb{O}}}}).$$

$||A|| = ||Y||^2$, $\tau_{\mathbb{O}}(X,Y) = A$ and the factor $e^{-\sqrt{2}\pi||Y||} =< \mathbf{s}_0, \mathbf{t}_0 >$, where $\mathbf{s}_0$ is a unitary trivialization of $\mathbb{L}$.

Main theorem

(1) Let $\varepsilon = -\frac{47}{4}$, then the Bargmann type transformation $\mathfrak{B} : \mathcal{F}_{-47/4} \longrightarrow L_2(P^2\mathbb{O}, dv_{P^2\mathbb{O}})$ is an isomorphism, although it is not unitary.

(2) If $-22 < \varepsilon < -\frac{47}{4}$, then the inverse of the Bargmann type transformation

$$\mathfrak{B}^{-1} : L_2(P^2\mathbb{O}, dv_{P^2\mathbb{O}}) \longrightarrow \mathcal{F}_\varepsilon$$

is bounded, but and the Bargmann type transformation can not be extended to the whole Fock-like space $\mathcal{F}_\varepsilon$.

continued

(3) If $\varepsilon > -\frac{47}{4}$, then the Bargmann type transformation is bounded with the dense image, but not an ismorphism between the spaces $\mathcal{F}_\varepsilon$ and $L_2(P^2\mathbb{O}, dv_{P^2\mathbb{O}})$.

(4) Let $\varepsilon \leq -22$. Then, for such a k that $4k + 44 + 2\varepsilon \leq 0$, the integral defining the norm of the polynomials of such degree does not converge, although the Bargmann type transformation can be defined for such polynomials. Hence by defining an inner product on the finite dimensional space $\sum_{4k+44+2\varepsilon\leq 0} \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$ in a suitable way, the Bargmann type transformation behave in the same way as the case of (2).

Outline of a proof of the main theorem

Fix the notations of polynomials.

- $\mathbb{C}[\mathcal{J}(3)] = \sum \mathcal{P}_k[\mathcal{J}(3)]$:complex coefficient polynomial (functions) on $\mathcal{J}(3)$, that is polynomials of 27 real variables.

- $\mathcal{P}_k[P^2\mathbb{O}]$:restrictions of homogeneous polynomials $\mathcal{P}_k[\mathcal{J}(3)]$ to $P^2\mathbb{O}$.

- $\mathbb{C}[\mathbb{C} \otimes \mathcal{J}(3)] = \sum \mathcal{P}_k[\mathbb{C} \otimes \mathcal{J}(3)]$:complex coefficient polynomials of complex 27 variables

- $\mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$:restrictions of homogeneous polynomials $\mathcal{P}_k[\mathbb{C} \otimes \mathcal{J}(3)]$ to $\mathbb{X}_{\mathbb{O}}$.

- The action of F_4 is extended to both of the polynomial algebras in the natural way.

[I]

By the fundamental theorem of compact symmetric spaces, $L_2(P^2\mathbb{O}, dv_{P^2\mathbb{O}})$ is decomposed into irreducible subspaces

$$L_2(P^2\mathbb{O}, dv_{P^2\mathbb{O}}) = \overline{\sum_{k=0}^{\infty} \oplus E_k},$$

where each E_k is an irreducible representation space of the group F_4 (we may put $\dim E_k < \dim E_{k+1}$) and appears with the multiplicity “one”.

It holds $\sum_{k=0}^{\infty} \mathcal{P}_k[P^2\mathbb{O}] = \sum_{k=0}^{\infty} \oplus E_k$, more presicely for each $k \geq 0$,

$$\mathcal{P}_k[P^2\mathbb{O}] = \sum_{\ell=0}^{k} \oplus E_\ell \,.$$

This is not enough for the analysis of the Bargmann type transformation. We need the irreducible decomposition of the space $\mathcal{P}_k[\mathcal{J}(3)]$ realizing the spaces E_k.

There are three ways for the realization of E_k. I explain two of them, one has a similar aspect to the harmonic plynomials and one is constructive. For the irreducibility we need all of them.

[II]

- $I_k \subset \mathcal{P}_k$: invariant polynomials by the action of F_4.

Then by the

Lemma

For any matrix $X \in \mathcal{J}(3)$, there exists $\alpha \in F_4$ such that $\alpha(X)$ is a diagonal matrix and the set of the (complex) numbers appearing in the diagonal does not depend on such $\alpha \in F_4$.

$\boxed{T_1(X) = \mathrm{tr}(X)}$, $\boxed{T_2(X) = \mathrm{tr}(X^2)}$ and $\boxed{T_3(X) = \mathrm{tr}(X^3)}$

are the generators of the subalgebra $I := \sum_{k=0}^{\infty} I_k$ of invariant polynomials.

Example: $I_4 = [\{T_3T_1, {T_2}^2, T_2{T_1}^2, {T_1}^4\}]$.

Let $D_x = \sum a_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ be a constant coefficient differential operator on $\mathcal{J}(3)$ (on Euclidean space), then the relation

$$e^{-<x,\xi>} D_x(e^{<x,\xi>}) = \sum a_\alpha \xi^\alpha := P_D(\xi)$$

gives an isomorphism between the polynomial algebra and the algebra of constant coefficient differential operators.

Then the differential operators corresponding to *"invariant polynomials"* are the *"invariant differential operators"*. Especially we denote the differential operators corresponding to the generators $\{T_i\}$ by $L \leftrightarrow T_1$, $\Delta \leftrightarrow T_2$ and $\Gamma \leftrightarrow T_3$ and their explicit expressions are given as follows:

- $L := \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i}$,
- $\Delta = \sum \frac{\partial^2}{\partial {\xi_i}^2} + 2 \sum_{i=0}^{7} \left(\frac{\partial^2}{\partial {x_i}^2} + \frac{\partial^2}{\partial {y_i}^2} + \frac{\partial^2}{\partial {z_i}^2} \right)$,
- $\Gamma := \sum \frac{\partial^3}{\partial {\xi_j}^3} + 3 \left(\frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) \circ \sum_{i=0}^{7} \frac{\partial^2}{\partial {z_i}^2}$

$$+ 3 \left(\frac{\partial}{\partial \xi_3} + \frac{\partial}{\partial \xi_1} \right) \circ \sum_{i=0}^{7} \frac{\partial^2}{\partial {y_i}^2} + 3 \left(\frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial \xi_3} \right) \circ \sum_{i=0}^{7} \frac{\partial^2}{\partial {x_i}^2}$$

$$+ 6 \sum_{i,j,k=0}^{7} \pm \frac{\partial^3}{\partial x_i \partial y_j \partial z_k}.$$

Let
$\mathfrak{chp}_k = \{P \in \mathcal{P}_k[\mathcal{J}(3)] \mid L(P) = 0, \Delta(P) = 0, \Gamma(P) = 0\}$,
and we call a polynomial in $\mathfrak{chp}_k$ **Cayley harmonic polynomial**.

Let $\mathcal{H}_k$ be a subspace in $\mathcal{P}_k[\mathcal{J}(3)]$ generated by the functions of the form

$$\mathcal{J}(3) \ni X \longmapsto (\mathrm{tr}\,(X \circ A))^k, \; A \in \mathbb{X}_{\mathbb{O}}$$

Then

Proposition

$\mathfrak{cph}_k = \mathcal{H}_k \cong E_k$ and these are irreducible.

[III]

Let A_K be a transformation

$$A_k : H_k = \mathfrak{chp}_k \longrightarrow \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$$

defined by

$$A_k(h)(A) = \int_{P^2\mathbb{O}} h(X) \cdot \mathrm{tr}(X \circ A)^k dv_{P^2\mathbb{O}}, \ A \in \mathbb{X}_{\mathbb{O}}.$$

Then

Proposition

A_k is the inverse of the Bargmann transformation restricted to $\mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$.

[IV]
Determination of the constants

$$\mathfrak{B} \circ A_k = a_k Id$$

and their behavior when $k \to \infty$.
In fact, the constant is given as

$$a_k = \frac{Vol(S(P^2\mathbb{O}))}{2^{40+3\epsilon}\pi^{44+2\epsilon}} \frac{\Gamma(4k+44+2\epsilon)}{2^{8k}\pi^{4k} \cdot \dim H_k}$$

and by making use of asymptotic behaviour of some values of Γ-function we have the desired results.

Final remark

Remark

(1) In the cases of the original and other projective spaces, the parameter ϵ being zero case corresponds to our case of $\epsilon = -47/4$.

(2) If $\epsilon > -47/4$, then it means there are quantum states which can not be observed from classical way?

(3) It is clear $\mathcal{H}_k \subset \mathfrak{chp}_k$, but opposite is not.

(4) Fock spaces appearing here have reproducing kernel and they will have a relation with hypergeometric function.

(5) As an application of the characterization of the space $\mathbb{X}_{\mathbb{O}}$ by $A^2 = 0$, we can show concrete examples of compact Lagrangian submanifolds satisfying Maslov quantization condition in $T_0^*(P^2\mathbb{O}) \cong \mathbb{X}_{\mathbb{O}}$.

(6) I expcet that by the same method it will be possible to construct a simialr transformation on an exotic 7-sphere Σ^7_{GM}: called Gromoll-Meyer exotic 7-sphere.


Thank you for your Attention!