Semiclassical resonances generated by crossings of classical trajectories

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0. Aim

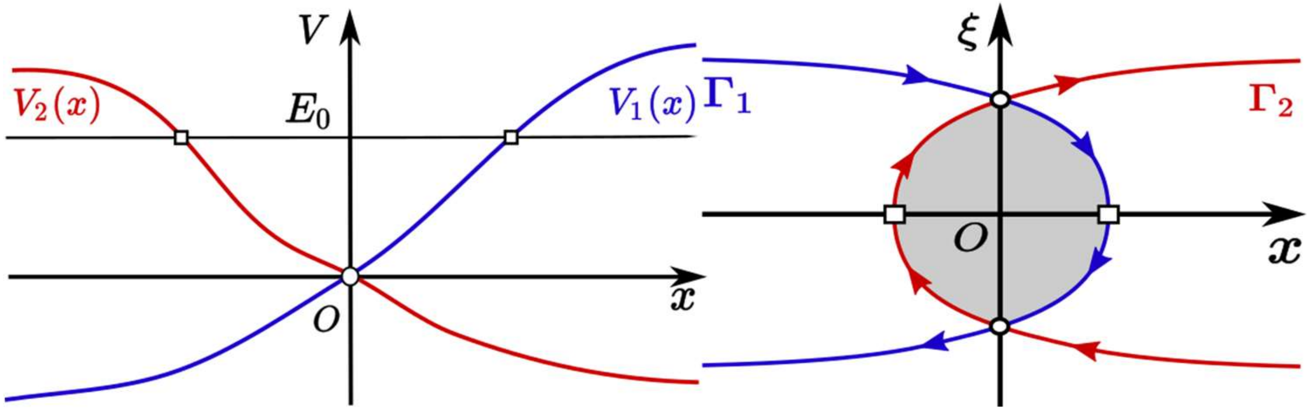
Consider a coupled 1D semiclassical Schrödinger operator $P(h)$ ($h > 0$ small):

$$P(h) = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix}, \quad P_j = -h^2 \frac{d^2}{dx^2} + V_j(x).$$

Q. Are there any resonances generated by crossings of two non-trapping classical trajectories?

A. -Yes!

We will give an example.



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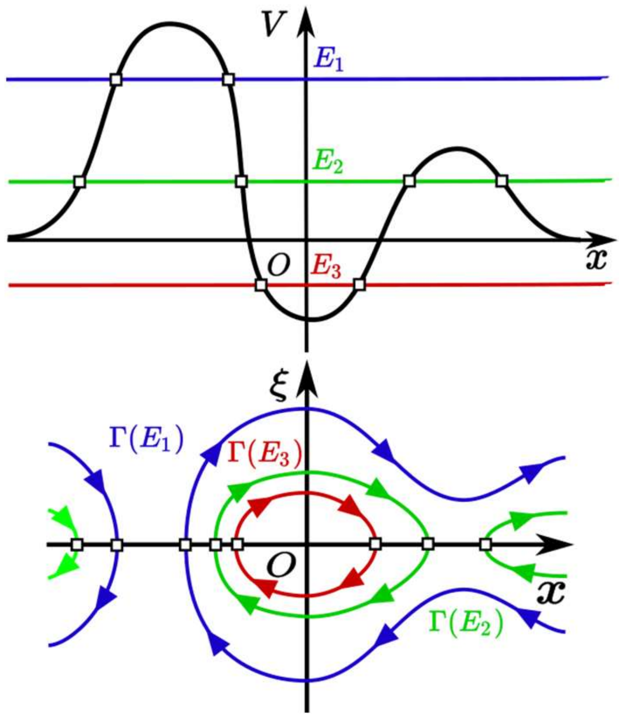
1. Backgrounds 1-1. non-trapping condition

- $p_j(x,\xi) = \xi^2 + V_j(x)$: the classical Hamiltonian associated to the (scalar) Schrödinger operator $P_j(h)$ $(j = 1,2)$
- $E \in \mathbb{R}$: non-trapping energy for p_j if $\left|\exp\left(tH_{p_j}\right)(x,\xi)\right| \to +\infty$ as $|t| \to +\infty$ for any $(x,\xi) \in p_j^{-1}(E) =: \Gamma_j(E)$.
- $\exp(tH_{p_j})(x,\xi)$: the Hamiltonian flow starts from $(x,\xi) \in T^*\mathbb{R}$ at the time $t = 0$.



An example of a potential $V(x)$ and energies E_1, E_2, E_3. Only E_1 is non-trapping

1-2. resonance free domain for scalar operators

(A1) $V_j \in C^\infty(\mathbb{R};\mathbb{R})$ can be extended to a bounded holomorphic function on

$$\Sigma = \{x \in \mathbb{C}\,;\ |\mathrm{Im}\,x| < (\tan\theta_0)\langle \mathrm{Re}\,x\rangle\} \text{ for some } 0 < \theta_0 < \pi/2.$$

$\mathrm{Res}(P_j)$: the set of resonances of P_j , that is, eigenvalues of

$$U_\theta\ P_j(h)U_\theta^{-1} = e^{-2i\theta}h^2D_x^2 + V_j(xe^{i\theta})\,,\quad U_\theta u(x) = u(xe^{i\theta}).$$

<u>Theorem</u> (Helffer-Sjöstrand 1986)

(A1) and E_0 is a non-trapping energy for p_j

$$\Longrightarrow\ \exists\varepsilon > 0 \text{ s.t. } \mathrm{Res}(P_j) \cap B_{\mathbb{C}_-}(E_0,\varepsilon) = \emptyset.$$

2. Result 2-1. assumptions

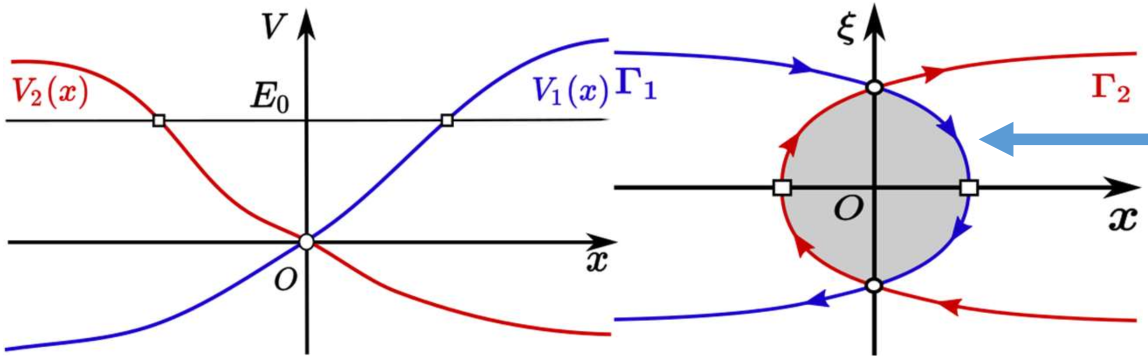
$$P(h) = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix}, \quad P_j = -h^2 \frac{d^2}{dx^2} + V_j(x).$$

(A2) $\exists v_j^{\pm} \coloneqq \lim_{\mathrm{Re}\, x \to \pm\infty} V_j(x)$ satisfying $v_1^- < E_0 < v_1^+,\ v_2^- > E_0 > v_2^+$.

$V_j^{-1}(E_0) = \{x_j\}$ (singleton) for each $j = 1{,}2$. (E_0: non-trapping for p_j)

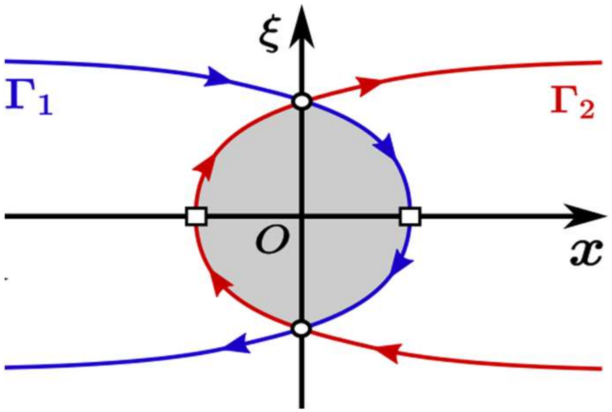
(A3) $V_1 = V_2$ only at $x = 0$. $V_1(0) = V_2(0) = 0$, $V_1'(0) > V_2'(0)$.

(A4) $W(x, hD_x) = r_0(x) + ir_1(x)hD_x$, where $r_0, r_1 \in C^\infty(\mathbb{R}; \mathbb{R})$ can be extended holomorphically to Σ. $\big(r_0(0), r_1(0)\big) \neq (0{,}0)$.



This behaves like a periodic trajectory!

2-2. Existence and asymptotics of resonances



γ: “periodic trajectory” with period $T = S'(E_0)$.

$S(E) = \int_{x_1}^{0} \sqrt{E - V_1(x)}dx + \int_0^{x_2} \sqrt{E - V_2(x)}dx$: action of γ.

$\mathrm{Res}_0(P) = \{E \in \mathbb{C}; C_0(E;h) = 1\}$: “pseudo-resonances.”

$C_0(E;h) = \frac{ihe^{iS(E)/h}\pi W_0}{\left(V_1'(0)-V_2'(0)\right)\sqrt{E_0}}$, $W_0 = r_0(0) + ir_1(0)\sqrt{E_0}$.

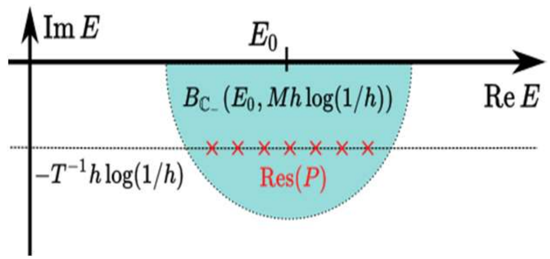
<u>Theorem</u> (H 2021)

Under (A1-4), $\mathrm{Res}\,(P) \cap B_{\mathbb{C}_-}\left(E_0, Mh\log(1/h)\right) \neq \emptyset$ for $M > T^{-1}$.

Moreover, $\mathrm{dist}\left(\mathrm{Res}(P), \mathrm{Res}_0(P)\right) = o(h)$ in $B_{\mathbb{C}_-}\left(E_0, Mh\log(1/h)\right)$.

<u>Remark</u> For $E \in B_{\mathbb{C}_-}(E_0, Mh\log(1/h))$, $E \in \mathrm{Res}_0(P) \Leftrightarrow \exists n \in \mathbb{N}$ s.t.

$$S(E) = h[(2n+1)\pi - \arg W_0] - ih\left(\log\left(\frac{1}{h}\right) + \log\left(\frac{\pi|W_0|}{(V_1'(0) - V_2'(0))\sqrt{E_0}}\right)\right).$$



2-3. remarks

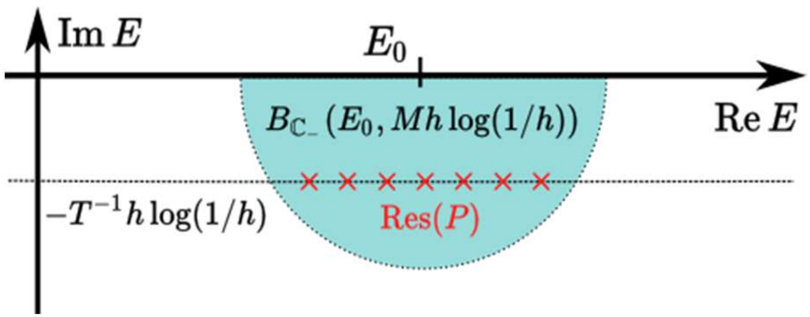
1. The assumptions can be relaxed as follows:
   - holomorphic on Σ $\mapsto$ on $\Sigma \cap \{|\mathrm{Re}\, x| > R_0\} \cup V_j^{-1}(E_0)$ for some R_0 >0.
   - $V_1 = V_2$ only at $x = 0$ $\mapsto$ at finitely many points (and $V_1' \neq V_2'$ there).
2. This theorem optimizes the previous result (H 2020):

Under (A1-4), $\mathrm{Res}\,(P) \cap B_{\mathbb{C}_-}\big(E_0, Mh\log(1/h)\big) = \emptyset$ for $M < T^{-1}$.

<u>Theorem</u> (H 2021)

Under (A1-4), $\mathrm{Res}\,(P) \cap B_{\mathbb{C}_-}\big(E_0, Mh\log(1/h)\big) \neq \emptyset$ for $M > T^{-1}$.

Moreover, $\mathrm{dist}\,\big(\mathrm{Res}(P), \mathrm{Res}_0(P)\big) = o(h)$ in $B_{\mathbb{C}_-}\big(E_0, Mh\log(1/h)\big)$.

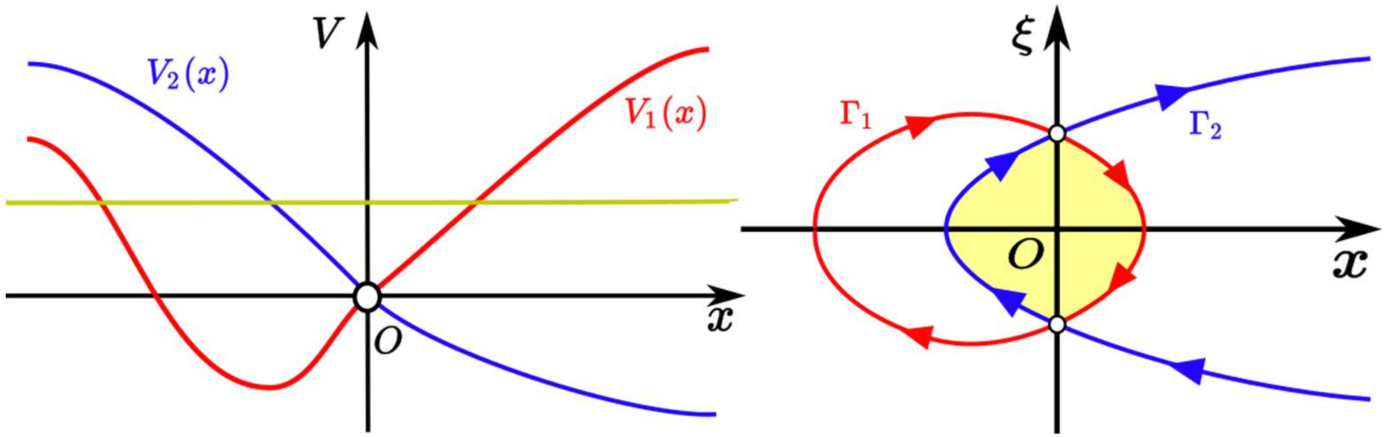


<u>Quantization Condition</u>

$\forall E \in \mathrm{Res}\,(\mathrm{P}) \cap B_{\mathbb{C}_-}\big(E_0, Mh\log(1/h)\big), \exists n \in \mathbb{N}$ s.t. $S(E)/h$
$= (2n+1)\pi - \arg W_0 + O(h^{1/2}) - i\,(\log(1/h) + O(1))$.

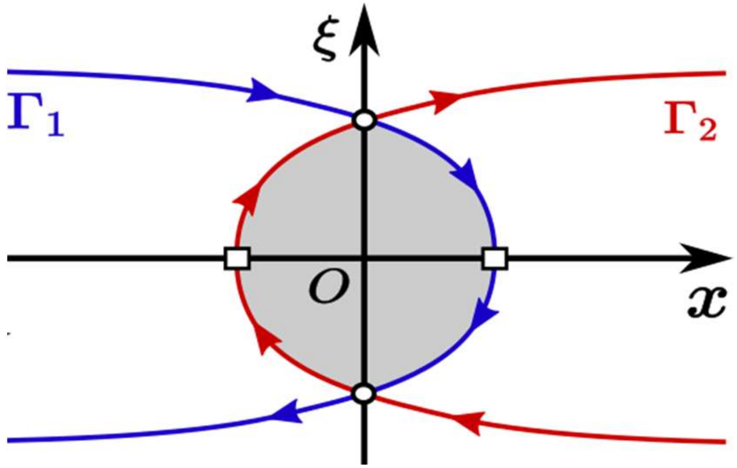
2-4. related work

- Resonances of coupled Schrödinger operators:
  Martinez (1994), Nakamura (1995), Baklouti (1998), Ashida (2018), etc.
- Resonances in the presence of crossings of classical trajectories:
  Fujiié-Martinez-Watanabe (2016,2017,2021).



E_0 is trapping for p_1, non-trapping for p_2 (Fujiié-Martinez-Watanabe 2021)
[Resonance of $P(h)$]$\sim$ [Eigenvalue of $P_1(h)$] $-ih^2$[in terms of $(S(E), W_0)$].

3. Method 3-1. outline

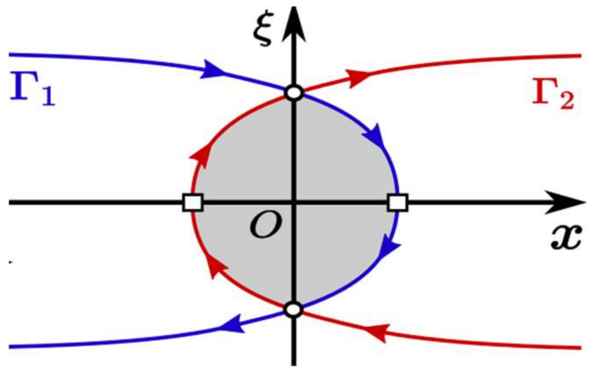


We prove the theorem in 4 steps.

1. Apply the microlocal connection formulae due to Fujiié-Martinez-Watanabe (2021) to the microlocal Cauchy problem along γ.
2. A contradiction argument on monodromy shows a resolvent estimate $\|(P_\theta - E)^{-1}\| \leq h^{-N}$ for E; $\mathrm{dist}\big(E, \mathrm{Res}_0(P)\big) > \varepsilon h$.
3. Let $\tilde{E} \in \mathrm{Res}_0(P)$. We obtain $(P_\theta - E)^{-1}v = (1 - C)^{-1}w + O(h^\infty)$ on $\partial B\big(\tilde{E}, \varepsilon h\big)$ with v, w, C analytically depending on E.
4. Since the RHS $(1 - C)^{-1}w$ has a unique simple pole in $B\big(\tilde{E}, \varepsilon h\big)$, so does $(P_\theta - E)^{-1}$ (by the residue theorem), that is, a resonance of $P(h)$.

Note Steps 2-4 are applications of the one of Bony-Fujiié-Ramond-Zerzeri(2019).

3-2. microlocal terminologies

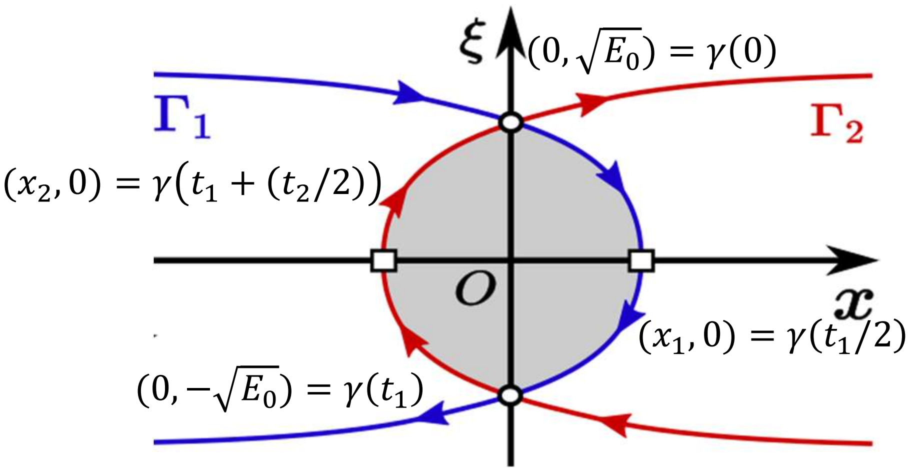


1. For $a \in C_0^\infty(T^*\mathbb{R})$, define $a^w(x, hD_x)$ on $C_0^\infty(\mathbb{R})$ by
$$a^w(x, hD_x)u(x) := \frac{1}{2\pi h}\int_{\mathbb{R}^2} e^{i(x-y)\xi/h} a\left(\frac{x+y}{2}, \xi\right) u(y)dyd\xi\ .$$
2. For $f \in L^2$ with $\|f\|_{L^2} \leq 1$, say $f \sim 0$ microlocally near (x_0, ξ_0) if $\exists \chi \in C_0^\infty(T^*\mathbb{R})$ s.t. $\chi(x_0, \xi_0) \neq 0, \|\chi^w(x, hD_x)f\|_{L^2} = O(h^\infty)$.
3. $\|u\|_{L^2} \leq 1, \|(P(h) - E)u\|_{L^2} = O(h^\infty)$
$\Longrightarrow u \sim 0$ microlocally on $T^*\mathbb{R} \setminus (\Gamma_1 \cup \Gamma_2)$.
4. On each connected component of $(\Gamma_1 \cup \Gamma_2) \setminus \{(0, \pm\sqrt{E_0}), (x_j, 0)\}$, the space of microlocal solutions is spanned by a WKB-type solution.

3-3. microlocal Cauchy problem (1)

Put $\gamma(t) = \begin{cases} \exp(tH_{p_1})(0,\sqrt{E_0}),\ t \in [0,t_1] \\ \exp\left((t-t_1)H_{p_2}\right)(0,-\sqrt{E_0}),\ t \in [t_1,T] \end{cases}$, $\gamma(t+T) = \gamma(t)$,

where $t_j = s_j'(E_0)$, $s_j(E) = \left|\int_0^{x_j(E)} \sqrt{E - V_j(x)}\,dx\right|$, $T = t_1 + t_2$.



On $\gamma(\,]0,t_1/2[\,)$, the space of microlocal solutions is spanned by a WKB-type solution

$$w(x,h) \sim e^{i\int_0^x \sqrt{E-V_1(y)}dy/h} \begin{pmatrix} \sigma_1(x,h) \\ h\sigma_2(x,h) \end{pmatrix},$$

$$\sigma_1 = \frac{1+O(h)}{(E-V_1)^{1/4}},$$

$$\sigma_2 = \frac{r_0(x) - ir_1(x)\sqrt{E-V_1(x)}}{(V_1(x)-V_2(x))(E-V_1)^{1/4}}(1+O(h)).$$

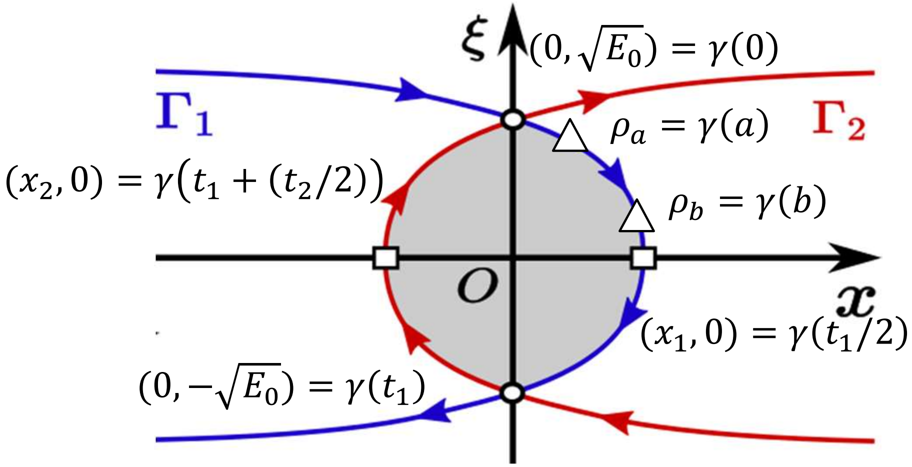
3-3. microlocal Cauchy problem (2)

Take $0 < a < b < t_1/2$ and put $\rho_a = \gamma(a)$, $\rho_b = \gamma(b)$.
Continue microlocal solution near ρ_b to ρ_a in two directions (positive and negative).
(negative) $(P - E)u \sim 0$ near $\gamma([a, b])$, $u \sim \alpha w$ near $\rho_b \Longrightarrow u \sim \alpha w$ near ρ_a.

<u>Lemma</u> (positive direction)

$(P - E)u \sim 0$ near $\gamma([b, T + a])$, $u \sim 0$ near Γ_{in}, $u \sim \alpha_b w$ near ρ_a

$\Longrightarrow u \sim \alpha_a w$ near ρ_a with $\alpha_a = C\alpha_b,\ C = C_0\left(1 + O\left(h^{1/2}\right)\right)$.



<u>Recall</u> $C_0(E; h) = \dfrac{ihe^{iS(E)/h}\pi W_0}{\left(V_1'(0) - V_2'(0)\right)\sqrt{E_0}}$.

Continue given microlocal solution by

- Along each path: propagation of singularities
- Turning points □: Maslov's theory
- Crossing points ○: reduction to a normal form (Fujiié-Martinez-Watanabe: 2021)

3-4. Resolvent estimate

Proposition

$\|(P_\theta - E)^{-1}\| \leq h^{-N}$ on $B_{\mathbb{C}_-}\big(E_0, Mh\log(1/h)\big) \setminus \big(\mathrm{Res}_0(P) + B(0, \varepsilon h)\big)$.

In particular, there is no resonances.

Proof by a contradiction argument

1. Suppose $\exists (h_k, E_k, u_k)_{k\geq 0}$ such that $h_k \to 0_+$ as $k \to +\infty$, $(P_\theta(h_k) - E_k)u_k = O(h_k^\infty)$, $\|u_k\|_{L^2} = 1$.

2. By Lemma (microlocal Cauchy problem), we have $u_k \sim C(E;h)u_k$.

3. Then, $|1 - C(E;h)| \geq \exists c > 0$ on $B_{\mathbb{C}_-}\big(E_0, Mh\log(1/h)\big) \setminus \big(\mathrm{Res}_0(P) + B(0, \varepsilon h)\big)$ implies $\|u_k\|_{L^2} = O(h^\infty)$ (contradiction!).

Remark We apply the localization theorems due to Bony-Fujiié-Ramond-Zerzeri (2019) in the step 3.

3-5. Inhomogeneous Cauchy problem

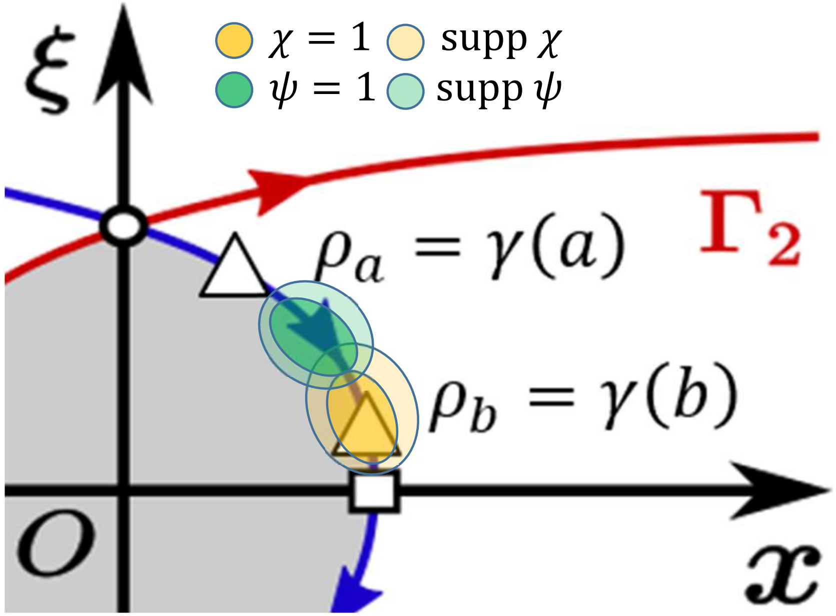
<u>Proposition 2</u> $(P_\theta - E)^{-1}v \sim (1-C)^{-1}w$ near ρ_a for $E \in \partial B(\tilde{E}, \varepsilon h)$, where w: WKB function (given in 3-3), $\tilde{E} \in \mathrm{Res}_0(P)$, $v = \psi^w[P, \chi^w]w$, $\psi, \chi \in C_0^\infty(T^*\mathbb{R})$ supported near $\gamma(\,]0, t_1/2[\,)$.

1. $u_p \coloneqq \chi^w w$ satisfies

$$\begin{cases} (P-E)u_p \sim v \text{ near } \gamma([b, T+a]), \\ u_p \sim 0 \text{ near } \rho_b,\ \ u_p \sim w \text{ near } \rho_a. \end{cases}$$

2. Put $u \coloneqq (P_\theta - E)^{-1}v$, $u_{\mathrm{hom}} \coloneqq u - u_p$. Then $u_{\mathrm{hom}} \sim \alpha_b C w$ near ρ_a, where $u_{\mathrm{hom}} \sim \alpha_b w$ near ρ_b (by Lemma).

3. $u = u_{\mathrm{hom}} + u_p \sim Cu + w$ near ρ_a with $u = (P_\theta - E)^{-1}v$ shows Proposition 2.



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