Localization of the ground state of the Nelson model

Fumio Hiroshima
Kyushu university, Japan

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Plan of Talks

1. **Spatial localization in QM by FKF**
2. **Renormalized Nelson Hamiltonian**
3. **Existence of the ground state φ_g**
   - FKF for the renormalized Nelson model
   - Existence of the ground state φ_g
4. **Localizations**
   - Spatial localization
   - Renormalized Gibbs measures
5. **Concluding remarks**

Let us consider the Schrödinger equation:

$$hf = Ef, \quad h = -\frac{1}{2}\Delta + V.$$

Spatial decays of f

(1) L^2-estimate: $\|e^{c|x|}f\| < \infty$
(2) point-wise-estimate: $a_1 e^{-c|x|} \le |f(x)| \le a_2 e^{-c|x|}$ a.e. $x \in \mathbb{R}^3$.

We are interested in "point-wise-estimate" from both upper and lower.
Roughly speaking solving the equation ($d=1$)

$$-\frac{1}{2}(e^{-r})'' + Ve^{-r} = Ee^{-r}$$

we obtain $(r')^2 = 2(V-E)$, $r(x) = \int^x \sqrt{2(V(x)-E)}dx \sim |x|\sqrt{2(V(x)-E)}$.

$$|f(x)| \sim Ae^{-c|x|\sqrt{V(x)}}.$$

FKF

$$(e^{-th}f)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)ds} f(B_t)],\ a.e. x \in \mathbb{R}^3$$

►$(B_t)_{t\geq 0}$ is 3-dim Brownian motion with $B_0 = x$ on $(\Omega, \mathscr{Y}, W^x)$.
$\mathbb{E}^x[\ldots] = \int_\Omega \ldots dW^x$.

Ground tate $h\varphi_{\rm p} = E\varphi_{\rm p}$**,** $E = \inf Spec(h)$**.**

$$\frac{e^{-th}f}{\|e^{-th}f\|} \to \varphi_{\rm p}\ (t \to \infty) \text{ in } L^2,\ (f \geq 0)$$

►We keep in mind that

$$\varphi_{\rm p}(x) \sim \frac{(e^{-th}f)(x)}{\|e^{-th}f\|} = \frac{\mathbb{E}^x[e^{-\int_0^t V(B_s)ds} f(B_t)]}{(\int |\mathbb{E}^x[e^{-\int_0^t V(B_s)ds} f(B_t)]|^2 dx)^{1/2}} \quad t \gg 1$$

Examples $h=-\frac{1}{2}\Delta+\frac{1}{2}|x|^2$

►(1) Set $hf=Ef$. Then $|x|\sqrt{V(x)}=|x|^2$ and it is known that

$$f(x)=h(x)e^{-|x|^2/2}.$$

►(2) Set $f=1\!\!1\notin L^2$. Then

$$\begin{aligned}\frac{(e^{-th}1\!\!1)(x)}{\|e^{-th}1\!\!1\|} &= \frac{\mathbb{E}^x[e^{-\frac{1}{2}\int_0^t|B_s|^2ds}]}{(\int|\mathbb{E}^x[e^{-\frac{1}{2}\int_0^t|B_s|^2ds}]|^2dx)^{1/2}}\\ &= \frac{e^{-|x|^2/2\coth t}}{(\int e^{-|x|^2/\coth t}dx)^{1/2}} \quad \text{(Yor formula)}\\ &\stackrel{t\to\infty}{\Longrightarrow} \pi^{-3/4}e^{-|x|^2/2} \quad \text{(ground state)}\end{aligned}$$

Let $V^-(x) = \inf\{V(y) \mid |x-y| \le 1\}$ and $hf = Ef$.

$$f(x) = e^{-t(h-E)} f(x) = \mathbb{E}^x[e^{-\int_0^t (V(B_s)-E)ds} f(B_t)]$$

Upper bound (Carmona 79)

$$|f(x)| \le a e^{-c|x|\sqrt{V_-(x)-E}}$$

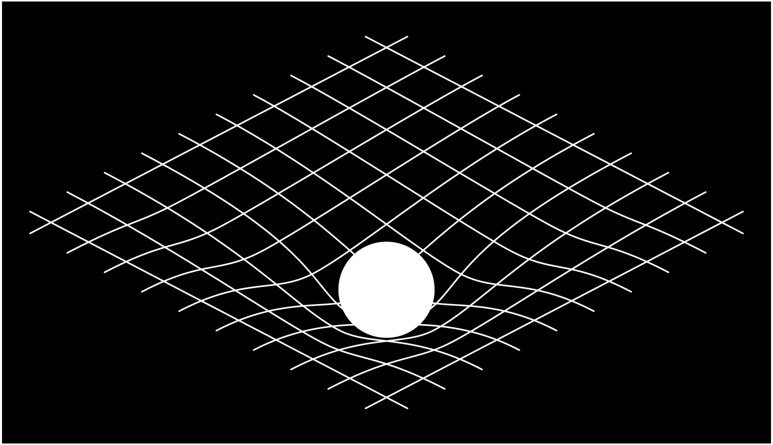
Let φ_p be the ground state. Let $K = [-b_1, b_1] \times [-b_2, b_2] \times [-b_3, b_3]$.

$$\inf_{x \in K} \varphi_p(x) > \exists \varepsilon_K > 0.$$

※ If $V = V_+ - V_-$ and V_+ is local-Kato, then $x \mapsto \varphi_p(x)$ is cont due to the smoothing effect of $e^{t\Delta}$. $V_a(x) = \sup\{V(y) \mid |x_j - y_j| \le 1 + |x_j|, j = 1, 2, 3\}$

Lower bound (Carmona 79)

$$|\varphi_p(x)| \ge a \varepsilon_K e^{-c|x|\sqrt{V_a(x)-E}}.$$



$$H_{\varepsilon,\Lambda} = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + g\phi(\rho_\Lambda(\cdot - x)) - E_\Lambda$$

Gaussian random variable $\phi(f), (f \in L^2_{\mathbb{R}}(\mathbb{R}^3))$ on (Q, Σ, μ)

$$\mathbb{E}_\mu[\phi(f)] = 0, \quad \mathbb{E}_\mu[\phi(f)\phi(g)] = \frac{1}{2}(f, g)_2$$

※ $\mathbb{E}_\mu[\ldots] = \int_Q [\ldots] d\mu$ and $\mathbb{E}_\mu[e^{z\phi(f)}] = e^{\frac{z^2}{4}\|f\|_2^2}$

Boson Fock space

$$\mathscr{F} = L^2(Q) = \overline{LH\{:\phi(f_1)\cdots\phi(f_n):\}}$$

►$H_{\rm f} = d\Gamma(\hat{\omega})$ is "the free field Hamiltonian" defined by

$$H_{\rm f} : \phi(f_1)\cdots\phi(f_n) := \sum_j : \phi(f_1)\cdots\phi(\hat{\omega} f_j)\cdots\phi(f_n) :$$

$$H_{\rm f} \mathbb{1} = 0$$

where $\hat{\omega} = \omega(-i\nabla)$ with

$$\omega(k) = \begin{cases} |k| & \textit{massless} \\ \sqrt{|k|^2 + \mu^2} & \textit{massive} \end{cases}$$

Total Hilbert space

$$\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F} \cong L^2(\mathbb{R}^3_x \times Q_\phi)$$

※ $F = F(x, \phi) \in \mathscr{H}$

Nelson Hamiltonian with UV cutoff Λ and IR cutoff ε

$$H_{\varepsilon,\Lambda} = H_S \otimes 1\!\!1 + 1\!\!1 \otimes H_{\mathrm{f}} + g\phi(\rho_\Lambda(\cdot - x)) - E_\Lambda$$

►$H_S = -\frac{1}{2}\Delta_x + V$

►$g \in \mathbb{R}$ coupling constant

►**Interaction**: $\phi(\rho_\Lambda(\cdot - x))$ Gaussian r.v. with test function $\rho_\Lambda(\cdot - x)$

►ρ_Λ has two parameters $0 \le \varepsilon < \Lambda < \infty$. IR cutoff ε. UV cutoff Λ.
The Fourier transform of $\rho_\Lambda(\cdot - x)$ is given by

$$\hat{\rho}_\Lambda(k,x) = \frac{e^{-ikx}}{\sqrt{\omega(k)}} \mathbb{1}_{\varepsilon \le |k| \le \Lambda} \quad 0 \le \varepsilon < \Lambda < \infty$$

►Removal of UV cutoff:

$$\lim_{\Lambda \to \infty} \hat{\rho}_\Lambda(k,x) = \hat{\rho}_\infty(k,x) = \frac{e^{-ikx}}{\sqrt{\omega(k)}} \mathbb{1}_{\varepsilon \le |k|} \notin L^2(\mathbb{R}^3)$$

Then $\phi(\rho_\infty(\cdot - x))$ is not well-defined as a Gaussian r.v.

Let $E_\Lambda = -\frac{1}{2}\int_{\mathbb{R}^3} \frac{1\!\!1_{\varepsilon\le|k|\le\Lambda}}{\omega(k)(\omega(k)+|k|^2/2)}dk(\to -\infty)$.

Thm (E. Nelson 64)

$$\exists \text{ s.a. } H_\infty > -\infty \text{ st } u-\lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda)} = e^{-tH_\infty}.$$

►The methods of renormalization of the Nelson Hamiltonian

- Operator theory (Nelson 64)
- Stochastic method (Gubinelli-FH-Lőrinczi14, Matte-Møller17)

FKF for $\Lambda = \infty$

Gubinelli-FH-Lőrinczi (JFA14), Matte-Møller(PTRF17)

Let $F, G \in \mathscr{H}$. Then

$$(F, e^{-tH_\infty} G) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[(F(B_0), e^{-\int_0^t V(B_s)ds} K_t G(B_t))_{\mathscr{F}} \right] dx.$$

Here the integral kernel is given by

$$K_t = e^S e^{a^\dagger(U_t)} e^{-tH_{\mathrm{f}}} e^{a(\bar{U}_t)} \ (a.s.\ bounded)$$

By FKF

$$\varphi_{\mathrm{g}}(x) = \mathbb{E}^x [e^{-\int_0^t (V(B_s)-E)ds} K_t \varphi_{\mathrm{g}}(B_t)] \quad a.e.(x, \phi) \in \mathbb{R}^3 \times Q.$$

φ_g: ground state $\Longleftrightarrow H\varphi_g = E\varphi_g$ with $E = \inf\sigma(H)$

	$\varepsilon > 0$	$\varepsilon = 0$
$\mu > 0$	*exist*	*exist*
$\mu = 0$	*exist*	*not exist*

Figure: ε IR cutoff, μ mass, Ground state for $\Lambda \leq \infty$

▶ $(\Lambda < \infty)$ Bach-Fröhlich-Sigal (AdvMath95), Arai-Hirokawa (JFA97), Spohn (LMP99), Gérard (AHP00), Griesemer-Lieb-Loss (InvMath01)

▶ $(\Lambda = \infty)$ $|g| \ll 1 \to$ Hirokawa-FH-Spohn (Adv Math 05)

$\forall g \to$ FH-Matte, preprint (19)

Upper bound (FH20)

Suppose that $V = W - U$ st $U \geq 0$ and $U \in L^p$ for $3/2 < p < \infty$, $W \in L^1_{loc}$ and $\inf_x W > -\infty$. Then

$$\|\varphi_g(x)\|_{\mathscr{F}} \leq A e^{-c|x|\sqrt{W_-(x)}}.$$

▶Idea $\|\varphi_g(x)\|_{\mathscr{F}} \geq (1\!\!1, \varphi_g(x))_{\mathscr{F}} \underset{a.e.x}{=} \mathbb{E}^x[e^{-\int_0^t (V(B_s)-E)ds}(1\!\!1, K_t \varphi_g(x))_{\mathscr{F}}]$

Lower bound (FH20)

Let $V = W - U$ st $U \geq 0$, $U \in L^p$ for $3/2 < p < \infty$, and W is local Kato. Then the map $\mathbb{R}^3 \ni x \mapsto (1\!\!1, \varphi_g(x))$ is continuous, and

$$\|\varphi_g(x)\|_{\mathscr{F}} \geq A e^{-c|x|\sqrt{W_a(x)}}.$$

►Let $0 \leq u \in L^2(\mathbb{R}^3)$ and since $(u \otimes 1\!\!1, \varphi_g) \neq 0$,

$$u_t = \frac{e^{-tH_\infty} u \otimes 1\!\!1}{\|e^{-tH_\infty} u \otimes 1\!\!1\|_{\mathscr{H}}} \stackrel{t\to\infty}{\to} \varphi_g \ \ strongly\ in\ \mathscr{H}$$

For $A : \mathscr{H} \to \mathscr{H}$, by FKF ∃a prob. measure μ_t on a path space st

$$(u_t, Au_t) = \mathbb{E}_{\mu_t}[f_A^t].$$

Hence

$$(\varphi_g, A\varphi_g) = \lim_{t\to\infty} \mathbb{E}_{\mu_t}[f_A^t].$$

Gibbs measure. FH AdvMath (14), FH-Matte (19)

$\exists \mu_\infty$ st $\mu_t \to \mu_\infty$ $(t \to \infty)$ in the local sense.

Let N be the number operator.

Super exp-decay of bosons FH-Matte(19)

$$(\varphi_{\rm g}, e^{+\beta N}\varphi_{\rm g}) = \mathbb{E}_{\mu_\infty}\left[e^{+(1-e^{+\beta})\int_0^\infty ds \int_{-\infty}^0 dr W(B_s-B_r, s-r)}\right] < \infty \quad \forall \beta \in \mathbb{R}.$$

Gaussian domination FH+Matte (19)

$$(\varphi_{\rm g}, e^{+\beta\phi(f)^2}\varphi_{\rm g}) = \frac{1}{\sqrt{1-\beta\|f\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{+\frac{\beta(\int_{-\infty}^\infty S(B_s,s)ds)^2}{(1-\beta\|f\|^2)}}\right] < \infty \quad \forall \beta < 1/\|f\|^2$$

and

$$\lim_{\beta\uparrow 1/\|f\|^2}(\varphi_{\rm g}, e^{\beta\phi(f)^2}\varphi_{\rm g}) = \infty$$

Cf. $h = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$ and $hf_n = Ef_n$. $\lim_{\beta\uparrow 1}(f_n, e^{\beta|x|^2}f_n) = \infty$.

Concluding remarks $\Lambda = \infty$

0. All results we obtain are independent of coupling constants.

1. Suppose that $\mu = 0$ (massless).
- Let $\varepsilon > 0$. Then H_∞ has the ground state.
- Let $\varepsilon = 0$. Then H_∞ has no ground state.

2. Properties of ground state:
- spatial decay: $e^{-|x|\sqrt{V(x)}} \leq \|\varphi_{\rm g}(x)\| \leq e^{-|x|\sqrt{V(x)}}$.
- super-exp. decay: $\|e^{\beta N}\varphi_{\rm g}\| < \infty$ for all $\beta \in \mathbb{R}$.
- Gaussian domination: $\lim_{\beta \uparrow 1/\|f\|^2} (\varphi_{\rm g}, e^{\beta\phi(f)^2}\varphi_{\rm g}) = \infty$

3.
- For decaying potential V we can also see a spatial decay.
- Agmon metric