Geometric analysis on manifolds with ends

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Alexander Grigory'an (Bielefeld)
and
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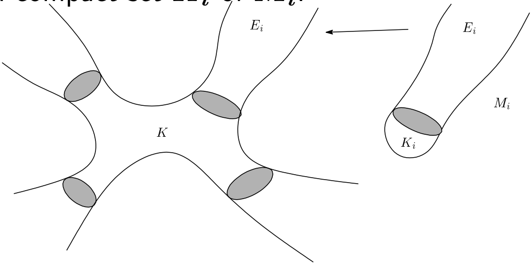
◦ Heat kernel estimates on connected sums of parabolic manifolds, J. Math. Pures Appl. 113(2018), 155-194.
◦ Geometric analysis on manifolds with ends, Advances in Analysis and Geometry, 3(2020), 325-343.
◦ Poincaré constant on manifolds with ends, in preparation.

Definition (Manifold with ends (connected sum))

Let M_1, $M_2, \ldots, M_k$ be geodesically complete, non-compact weighted manifolds of a same dimension.

We say that M is a manifold with k-ends $M_1, \ldots, M_k$ (and write $M = M_1 \# \cdots \# M_k$) if there exists a compact set $K \subset M$ so that $M \backslash K$ consists of k non-compact connected components $E_1, \ldots E_k$ so that each E_i is isometric to $M_i \backslash K_i$, exterior of a compact set K_i of M_i.



Our interest

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∘**Stable global structure under the quasi-isometric modification.**

In this talk, we focus on

- **Long time behavior of the heat kernel $p(t,x,y)$ on M**
- **We always assume that the heat kernel $p_i(t,x,y)$ on each end M_i satisfies the Li-Yau type estimates (two-sided matching Gaussian estimates):**

$$(LY) \quad p_i(t,x,y) \approx \frac{C}{V_i(x,\sqrt{t})} e^{-b d_i^2(x,y)/t}.$$

Typical examples

On $M = \mathbb{R}^n \# \mathbb{R}^n$, $n \geq 3$, for $x \in M_1, y \in M_2$

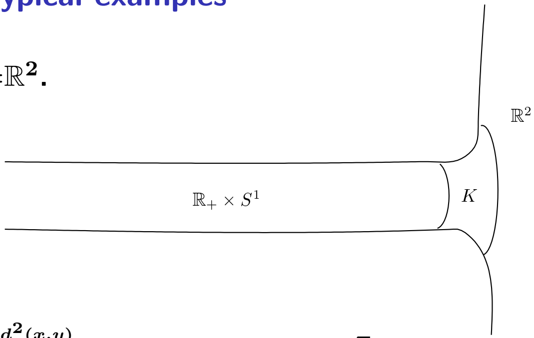
$$p(t,x,y) \approx \frac{1}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right) e^{-b\frac{d^2(x,y)}{t}}.$$

On $M = \mathbb{R}^2 \# \mathbb{R}^2$, for $x \in M_1, y \in M_2$,

$$p(t,x,y) \approx \begin{cases} \frac{1}{t \log^2 t} \left(\log t + \log^2 \sqrt{t} - \log|x| \log|y| \right) & \text{if } |x|,|y| \leq \sqrt{t}, \\ \frac{1}{t \log t} \log \frac{e\sqrt{t}}{|y|} e^{-b\frac{d^2(x,y)}{t}} & \text{if } |y| \leq \sqrt{t} < |x|, \\ \frac{1}{t \log t} \log \frac{e\sqrt{t}}{|x|} e^{-b\frac{d^2(x,y)}{t}} & \text{if } |x| \leq \sqrt{t} < |y|, \\ \frac{1}{t} \left(\frac{1}{\log|x|} + \frac{1}{\log|y|} \right) e^{-b\frac{d^2(x,y)}{t}} & \text{if } |x|,|y| > \sqrt{t}. \end{cases}$$

Typical examples

Let $M = (\mathbb{R}_+ \times S^1) \# \mathbb{R}^2$.



For $x \in M_1, y \in M_2$

$$p(t,x,y) \asymp \begin{cases} \frac{1}{t} e^{-b\frac{d^2(x,y)}{t}} & \text{if } |y| > \sqrt{t}, \\ \frac{1}{t}\left(1 + \frac{|x|}{\sqrt{t}} \log \frac{e\sqrt{t}}{|y|}\right) & \text{if } |x|, |y| \le \sqrt{t}, \\ \frac{1}{t} \log \frac{e\sqrt{t}}{|y|}\, e^{-b\frac{d^2(x,y)}{t}} & \text{if } |x| > \sqrt{t} \ge |y|. \end{cases}$$

History of geometric analysis on manifolds with ends

1979 **Kuz'menko, Molchanov**

Failure of Liouville property on $\mathbb{R}^n \# \mathbb{R}^n$, $n \geq 3$.

1991 **Grigor'yan, Saloff-Coste (with Moser, Kusuoka, Stroock etc)**

$$(LY) \iff (PI) + (VD) \iff (PHI)$$

1996 **Benjamini, Chavel, Feldman**

Bottleneck effect on the heat kernel on $\mathbb{R}^n \# \mathbb{R}^n$, $n \geq 3$:

For $x \in M_1$, $y \in M_2$ with $d(x, K) \approx d(y, K) \approx \sqrt{t}$,

$$p(t, x, y) \leq \frac{C}{t^{\frac{n}{2}+\epsilon}} \quad \text{for some } \epsilon > 0.$$

2002 **Grigor'yan and Saloff-Coste**
Estimate of the Dirichlet heat kernel exterior of a compact set

2002 **Grigor'yan and Saloff-Coste**
Estimate of the first hitting probability to a compact set

2009 **Grigor'yan and Saloff-Coste**
Heat kernel estimates on non-parabolic manifolds with ends.
Mixed case (i.e. parabolic # non-parabolic) is reduced to a non-parabolic case by using Doob's h-transform)

2018 **Grigor'yan, I and Saloff-Coste**
Heat kernel estimates on parabolic manifolds with ends.

Remark: M is called parabolic if the B.M. is recurrent.

Definition

A manifold M admits the Poincaré inequality if there exits $C, \kappa > 0$ such that for any $r > 0$, $x \in M$ and any $f \in C^1(B(x, r))$,

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leq Cr^2 \int_{B(x,\kappa r)} |\nabla f|^2 d\mu,$$

where $f_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu$.

The connected sum $M = M_1 \# M_2 = \mathbb{R}^n \# \mathbb{R}^n$ (n $\geq$ 3) doesn't satisfy the Poincaré inequality. Indeed, the function

$$f(x) = \begin{cases} 1 & x \in M_1, \\ -1 & x \in M_2 \end{cases}$$

implies that $f_{B(o,r)} = 0$ for a central point $o \in M$ and

$$\int_{B(o,r)} |f - f_{B(o,r)}|^2 d\mu \approx r^n, \quad \int_{B(o,\kappa r)} |\nabla f|^2 d\mu \approx const.$$

which fails (PI).

Heat kernel estimates

$p^D_{E_i}(t,x,y)$: **Dirichlet heat kernel on** $E_i = M_i \backslash K_i$,
τ_{E_i}: **First exit time to** $K_i \subset M_i$.
$\mathbb{P}_x(\tau_{E_i} \le t)$: **First exit probability from** x **to** K_i **by time** t.

Theorem 1 (Grigor'yan and Saloff-Coste, 2009)

Let $M = M_1 \# \cdots \# M_k$ *and fix* $o \in K$. *For* $x \in E_i$, $y \in E_j$,

$$p(t,x,y) \approx p^D_{E_i}(t,x,y) + p(t,o,o)\mathbb{P}_x(\tau_{E_i} \le t)\mathbb{P}_y(\tau_{E_j} \le t) +$$
$$\int_1^t p(s,o,o)ds\Big(\partial_t \mathbb{P}_x(\tau_{E_i} \le t)\mathbb{P}_y(\tau_{E_j} \le t) + \mathbb{P}_x(\tau_{E_i} \le t)\partial_t \mathbb{P}_y(\tau_{E_j} \le t)\Big).$$

Setting

We always assume that each end M_i satisfies (LY). If M_i is parabolic, we also assume that relatively connected annuli condition (RCA) :

Definition 2 (Relatively connected annuli condition (RCA))

M_i admits (RCA) for a fixed reference point $o_i \in M_i$ if there exists $A > 1$ and $r_0 > 0$ such that $\forall r > r_0$,

$$B_i(o_i, Ar) \backslash B_i(o_i, A^{-1}r)$$

is connected.

On-diagonal estimates: non-parabolic case

For a fixed reference point $o_i \in K_i \subset M_i$, let $V_i(r) = V_i(o_i, r) = \mu_i(B_i(o_i, r))$ and

$$h_i(r) := \int_1^r \frac{s ds}{V_i(s)}.$$

Remark: $h_i(r) \to \infty (r \to \infty) \iff M$ is parabolic.

Theorem 1 (Grigor'yan, Saloff-Coste (2009))

Let M be a non-parabolic manifold with ends $M_1, \ldots, M_k$. Then

$$p(t, o, o) \approx \frac{1}{\min_i V_i(\sqrt{t}) h_i^2(\sqrt{t})}.$$

On-diagonal estimates: parabolic case

Definition Let M_i be parabolic.
M_i is called subcritical if $h_i(s) \leq Cs^2/V_i(s)$ $(\forall s > 1)$.
M_i is called regular if there exist $\varepsilon, \eta > 0$ with $\eta < \varepsilon/2$ s.t.

$$c\left(\frac{t}{s}\right)^{\varepsilon} \leq \frac{V_i(t)}{V_i(s)} \leq C\left(\frac{t}{s}\right)^{2+\eta} \quad (\forall t > s > 0).$$

Example
$V_i(r) \approx r^{\alpha}(\log r)^{\beta}$ is parabolic if $\alpha < 2$ or $\alpha = 2, \beta \leq 1$.
V_i is subcritical if $\alpha < 2$ and regular if $\alpha = 2, \beta \leq 1$.

Definition
$V_m \succ V_i$ if for all $s > 0$

$$CV_m(s) \geq V_i(s) \text{ and } V_m(s)h_m^2(s) \leq CV_i(s)h_i^2(s).$$

On-diagonal estimates: parabolic case

Theorem 2 (Grigor'yan, I, Saloff-Coste, in preparation)

Let $M = M_1 \# \cdots \# M_k$ *is* parabolic. *Assume also that*

- *each end is either subcritical or regular.*
- *there exists* m *such that* $V_m \succ V_i$ *for all* $i = 1, \ldots, k$.

Then we have

$$p(t, o, o) \approx \frac{1}{V_m(\sqrt{t})}.$$

Examples

Let $M = \mathbb{R}^n \# \mathbb{R}^n$ $(n = 1, 2, \ldots)$**. Then**

$$p(t, o, o) \approx \frac{1}{t^{n/2}}.$$

Let $M = (\mathbb{R}_+ \times S^1) \# \mathbb{R}^2$**. Then**

$$p(t, o, o) \approx \frac{1}{t}.$$

Let $M = (\mathbb{R}^2 \times S^{n-2}) \# \mathbb{R}^n$ **with** $n \geq 3$**. Then** $h_1(r) \approx \log r$ **and**

$$p(t, o, o) \approx \frac{1}{t \log^2 t}.$$

Examples

Let $M = \mathcal{R}_1 \# \cdots \# \mathcal{R}_k$, **where** $\mathcal{R}_i$ **is a model manifold with** $V_i(o, r) \approx r^2 (\log r)^{\beta_i}$ **satisfying**

$$1 \geq \beta_1 \geq \beta_2 \cdots \geq \beta_k.$$

Note that all ends are parabolic and, especially, regular.

Then

$$p(t, o, o) \approx \frac{1}{t(\log t)^{\beta_1}}.$$

A highlight of the proof: non-parabolic case

Assume that $M_1, \dots M_k$ are non-parabolic.
They prove the Faber-Krahn inequality which implies that

$$\sup_{x \in M} p(t, x, x) \leq \frac{C}{V_{\min}(\sqrt{t})}.$$

Since all ends are non-parabolic, for all $i = 1, \dots, k$,

$$p(t, o, o) \geq c p^D_{E_i}(t, o_i, o_i) \geq c' p_i(t, o_i, o_i) \geq \frac{c''}{V_i(\sqrt{t})}.$$

A highlight of the proof: parabolic case

Assume that $M_1, \dots M_k$ are parabolic. We estimate the integrated resolvent kernel

$$\gamma_\lambda(x) := \int_K \int_0^\infty e^{-\lambda t} p(t,x,z)\,dz\,dt$$

and its derivative $\dot\gamma_\lambda$ in λ. They satisfy

$$\Delta\gamma_\lambda - \lambda\gamma_\lambda = -1_K, \quad \Delta\dot\gamma_\lambda - \lambda\dot\gamma_\lambda = -\gamma_\lambda.$$

In a open set $A \subset M$ with $K \subset A$,

$$(\Delta - \lambda)(\gamma_\lambda - \gamma_\lambda^A) = 0,$$

where $\gamma_\lambda^A(x) := \int_K \int_0^\infty e^{-\lambda t} p_A^D(t,x,z)\,dz\,dt$

We apply comparison principle (a sort of maximal principle).

On ∂A,

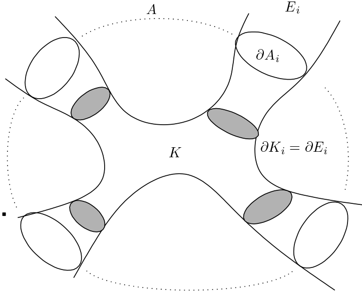
$$\gamma_\lambda - \gamma_\lambda^A \le \sum_i (\sup_{\partial A_i} \gamma_\lambda) 1_{\partial A_i}.$$

Then we obtain in A,

$$\gamma_\lambda - \gamma_\lambda^A \le \sum_i (\sup_{\partial A_i} \gamma_\lambda) h_i,$$

where h_i is the harmonic extension of $1_{\partial A_i}$.

$(\sum_i h_i \equiv 1)$



Since $(\Delta - \lambda)\gamma_\lambda = 0$ in E_i, using the comparison principle again on E_i, we obtain

$$\gamma_\lambda \le (\sup_{\partial E_i} \gamma_\lambda)(1 - \Phi_\lambda^{E_i}),$$

where

$$\Phi_\lambda^{E_i}(x) = \lambda \int_0^\infty e^{\lambda t} \mathbb{P}_x(\tau_{E_i} > t) dt.$$

By calculation,

$$\left(\sup_{\partial K} \gamma_\lambda\right) \sum_{i=1}^{k} \inf_{\partial A_i} \Phi_\lambda^{E_i} \le C.$$

Since

$$\inf_{\partial A_i} \Phi_\lambda^{E_i}(x) \ge \frac{c}{h_i(\frac{1}{\sqrt{\lambda}})},$$

We obtain

$$\sup_{\partial K} \gamma_\lambda \le C \min_i h_i\left(\frac{1}{\sqrt{\lambda}}\right).$$

By a similar argument, we obtain

$$(\sup_{\partial K}\dot{\gamma}_\lambda)\sum_{i=1}^{k}\inf_{\partial A_i}\Phi_\lambda^{E_i}\leq C+(\sup_{\partial K}\gamma_\lambda)\left(C+\sum_{i=1}^{k}\sup_{\partial A_i}\Psi_\lambda^{E_i}\right),$$

where

$$\Psi_\lambda^{E_i}(x)=\int_0^\infty te^{-\lambda t}\partial_t\mathbb{P}_x(\tau_{E_i}\leq t)dt.$$

By the assumption of M_i, we obtain

$$\sup_{\partial A_i}\Psi_\lambda^{E_i}\leq\frac{C}{\lambda^2 V_i(\frac{1}{\sqrt{\lambda}})h_i^2(\frac{1}{\sqrt{\lambda}})}$$

which concludes

$$\sup_{\partial K}\dot{\gamma}_\lambda\leq C\frac{\min_i h_i^2(\frac{1}{\sqrt{\lambda}})}{\lambda^2\min_i V_i(\frac{1}{\sqrt{\lambda}})h_i^2(\frac{1}{\sqrt{\lambda}})}=\frac{C}{\lambda^2\min_i V_i(\frac{1}{\sqrt{\lambda}})}.$$