Scaling limit for 1d Schrödinger operators with random decaying potential

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Decaying Potential Model

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$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on} \quad L^2(\mathbf{R})$$

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$$F \in C^\infty(M), \quad M : torus, \quad \langle F \rangle := \int_M F(x)dx = 0,$$
$$\{X_t\}_{t\in\mathbf{R}} : \text{ BM. on } M.$$

Known Results 1

(1) Spectrum (Kotani-Ushiroya, 1988)

$\sigma(H) \cap [0, \infty)$ is

$$\begin{cases} \text{a.c.} & (\alpha > 1/2) \\ \text{p.p. on } [0, E_c] \text{ and s.c. on } [E_c, \infty) & (\alpha = 1/2, \ \exists E_c) \\ \text{p.p.} & (\alpha < 1/2) \end{cases}$$

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Known Results 2

(3) Level statistics (Kotani-N14, N14, Kotani-N17)
$\{E_j(L)\}$: positive e.v.'s of $H_L := H|_{[0,L]}$, $E_0 > 0$: reference energy

$$\xi_{L,E_0} := \sum_j \delta_{L(\sqrt{E_j(L)}-\sqrt{E_0})}$$

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$$\text{Clock}(\theta) := \sum_{n\in\mathbf{Z}} \delta_{n\pi+\theta}, \quad \theta : \text{ r.v. on } [0,\pi)$$

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Remark for $\alpha = 1/2$

(1) $\tau(E_0)$: "Lyapunov exponent" in the sense that

$$H\varphi = E\varphi \Longrightarrow \varphi(x) \overset{|x|\to\infty}{\simeq} |x|^{-\tau(E)}.$$

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$$Sine(\beta) \overset{d}{\to} \begin{cases} Poisson(d\lambda/\pi) & (\beta \downarrow 0) \\ Clock(unif[0,\pi)) & (\beta \uparrow \infty) \end{cases}$$

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$\rightsquigarrow$ Sine (β) "interpolates" between Poisson and clock.

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Our aim $\quad \mathbf{Q} : \left(E^{(L)}_J, \mu^{(L)}_{E^{(L)}_J}\right) \stackrel{d}{\to} ?$

Known Results

(1) (N 07)
H : d-dimensional discrete random Schrödinger operator

$$J : \text{loc. region} \implies \left(E_J^{(L)}, \mu^{(L)}_{E_J^{(L)}} \right) \xrightarrow{d} \left(E_J, \delta_{unif[0,1]^d} \right)$$

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$\implies \lim_{L\to\infty} \mu^{(L)}_{E_J^{(L)}}$: given by an exponential Brownian motion.

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$\rightsquigarrow$ What about continuous and decaying potential case ?

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Theorem

$$\left(E_J^{(L)}, \mu^{(L)}_{E_j(L)}\right)$$
$$\stackrel{d}{\to} \begin{cases} \left(E_J, 1_{[0,1]}(t)dt\right) & (\alpha > 1/2) \\ \left(E_J, \dfrac{\exp\left(2\mathcal{Z}_{\tau(E_J)\log\frac{t}{U}} - 2\tau(E_J)\left|\log\frac{t}{U}\right|\right)dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_J)\log\frac{s}{U}} - 2\tau(E_J)\left|\log\frac{s}{U}\right|\right)ds}\right) & (\alpha = 1/2) \\ \left(E_J, \delta_U(dt)\right) & (\alpha < 1/2) \end{cases}$$

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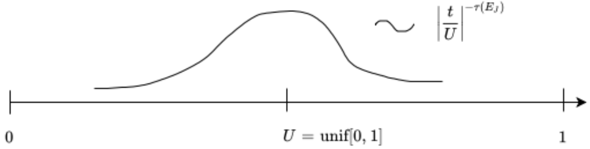

Figure: Limit Shape of eigenfunction

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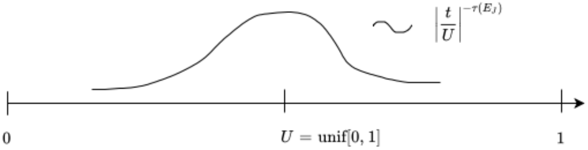

Figure: Limit Shape of eigenfunction

where

$$\begin{cases} U = \mathrm{unif}[0,1] & : \text{“loc. center”} \\ \tau(E_J) & : \text{strength of power law decay} \\ \mathcal{Z} & : \text{Brownian fluctuation} \end{cases}$$

Remark(cont)

Moreover, since

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eigenfunctions behave like

Figure: Limit Shape of eigenfunction Part 2

Sketch of Proof : Step 1

$$Hx_t \quad = \quad \kappa^2 x_t \overset{\text{Prüfer}}{\rightsquigarrow} \begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = r_t(\kappa) \begin{pmatrix} \sin\theta_t(\kappa) \\ \cos\theta_t(\kappa) \end{pmatrix}$$

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r_t(\kappa) &:= \exp(\rho_t(\kappa)) \rightsquigarrow \rho_t(\kappa) = \frac{1}{2\kappa} \mathrm{Im} \int_0^t e^{2i\theta_s(\kappa)} a(s) F(X_s) ds.
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$$\widetilde{\rho}_t^{(n)}(\kappa) := \rho_{nt}(\kappa) - \langle F g_\kappa \rangle \int_0^n a(s)^2 ds, \quad t \in [0,1],$$

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\end{aligned}
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Lemma 1 $\alpha = 1/2$, $\kappa_\lambda := \kappa_0 + \frac{\lambda}{n}$, $\kappa_0 := \sqrt{E_0}$,

$$
\Longrightarrow \quad \widetilde{\rho}_t^{(n)}(\kappa_\lambda) \overset{d}{\to} \exists \widetilde{\rho}_t(\lambda), \quad t \in [0,1], \text{ locally uniformly}
$$

$$
d\widetilde{\rho}_t(\kappa_\lambda) = \frac{\tau(\kappa_0^2)}{t} dt + \sqrt{\frac{\tau(\kappa_0^2)}{t}} dB_t^\lambda, \quad t > 0
$$

Proof : Step 2 : local version

Lemma 2 $\Xi^{(n)} := \sum_j \delta_{\left(n\left(\sqrt{E_j(n)} - \sqrt{E_0}\right), \mu^{(n)}_{E_j(n)} \right)} \xrightarrow{d} \Xi(E_0)$, where

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$$\Xi(E_0) := \begin{cases} \sum_{j \in \mathbf{Z}} \delta_{j\pi + \theta(E_0)} \otimes \delta_{1_{[0,1]}(t)dt} & (\alpha > 1/2) \\ \sum_{\lambda : Sine_{\beta(E_0)}} \delta_\lambda \otimes \delta\left(\frac{\exp(2\widetilde{\rho}_t(\lambda))dt}{\int_0^1 \exp(2\widetilde{\rho}_s(\lambda))ds} \right) & (\alpha = 1/2) \\ \sum_{j \in \mathbf{Z}} \delta_{P_j} \otimes \delta_{\delta_{\widetilde{P}_j}}, & (\alpha < 1/2) \end{cases}$$

where $\{P_j\} : Poisson(d\lambda/\pi)$, $\{\widetilde{P}_j\} : Poisson(1_{[0,1]}(t)dt)$.

Proof : Step 2 : local version

Lemma 2 $\Xi^{(n)} := \sum_j \delta_{\left(n\left(\sqrt{E_j(n)}-\sqrt{E_0}\right), \mu^{(n)}_{E_j(n)}\right)} \xrightarrow{d} \Xi(E_0)$, where

$$\Xi(E_0) \quad := \quad \begin{cases} \sum_{j\in\mathbf{Z}} \delta_{j\pi+\theta(E_0)} \otimes \delta_{1_{[0,1]}(t)dt} & (\alpha > 1/2) \\ \sum_{\lambda : Sine_{\beta(E_0)}} \delta_\lambda \otimes \delta\left(\frac{\exp(2\widetilde{\rho}_t(\lambda))dt}{\int_0^1 \exp(2\widetilde{\rho}_s(\lambda))ds} \right) & (\alpha = 1/2) \\ \sum_{j\in\mathbf{Z}} \delta_{P_j} \otimes \delta_{\delta_{\widetilde{P}_j}}, & (\alpha < 1/2) \end{cases}$$

where $\{P_j\} : Poisson(d\lambda/\pi)$, $\{\widetilde{P}_j\} : Poisson(1_{[0,1]}(t)dt)$.

$$\mathbf{E}\left[G(\lambda,\nu)d\Xi(\lambda,\nu)\right] \qquad \text{(intensity meas.)}$$

$$= \quad \frac{1}{\pi}\begin{cases} \int d\lambda \mathbf{E}\left[G\left(\lambda, 1_{[0,1]}(t)dt\right)\right] \\ \int d\lambda \mathbf{E}\left[G\left(\lambda, \frac{\exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{t}{U}} - 2\tau(E_0)\log\left|\frac{t}{U}\right|\right)dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{s}{U}} - 2\tau(E_0)\log\left|\frac{s}{U}\right|\right)ds}\right)\right] \\ \int d\lambda \mathbf{E}\left[G\left(\lambda, \delta_U\right)\right], \end{cases}$$

Step 3 : average on ref. energy

$$g_1(x) \ := \ (1-|x|)1(|x|\le 1), \quad g_2 \in C_b(\mathbf{R}\times\mathcal{P}(0,1)),$$

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g_1(x) &:= (1-|x|)1(|x|\le 1), \quad g_2 \in C_b(\mathbf{R}\times\mathcal{P}(0,1)), \\
G_L(E) &:= \sum_{E_j(L)\in J} g_1\left(L\left(\sqrt{E_j(L)}-\sqrt{E}\right)\right)\cdot g_2\left(E_j(L), \mu^{(L)}_{E_j(L)}\right)
\end{aligned}
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(1) Since $\int dN(E) g_1 = 1/(L\pi)$, we have

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\mathbf{E}\left[\int \frac{dN(E)}{N(J)} G_L(E)\right] = \frac{1}{N(J)}\frac{1}{\pi L}\mathbf{E}\left[\sum_{E_j(L)\in J} g_2\left(E_j(L), \mu^{(L)}_{E_j(L)}\right)\right]
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$$
= \mathbf{E}\Bigg[\frac{1}{\sharp\{\text{ eigenvalues of } H_L \text{ on } J \ \}(1+o(1))} \cdot \frac{1}{\pi} \cdot \sum_{E_j(L)\in J} g_2\left(E_j(L), \mu^{(L)}_{E_j(L)} \right) \Bigg]
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Step 3-2 : average on ref. energy

(2) Compute $\int \frac{dN(E)}{N(J)} \mathbf{E}[G_L(E)]$.

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&\sim \int \frac{dN(E)}{N(J)} \mathbf{E}\left[\int g_1(\lambda) g_2(E,\mu) d\Xi^{(L)}(\lambda,\mu) \right] \\
&\to \int \frac{dN(E)}{N(J)} \mathbf{E}\left[\int g_1(\lambda) g_2(E,\mu) d\Xi(\lambda,\mu) \right] \\
&= \int \frac{dN(E)}{N(J)} \frac{1}{\pi} \begin{cases} \int d\lambda \mathbf{E}\left[g_2\left(E, 1_{[0,1]}(t)dt\right)\right] \\ \int d\lambda \mathbf{E}\left[g_2\left(E, \frac{\exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{t}{U}} - 2\tau(E_0)\log\left|\frac{t}{U}\right| \right) dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{s}{U}} - 2\tau(E_0)\log\left|\frac{s}{U}\right| \right) ds} \right)\right] \\ \int d\lambda \mathbf{E}\left[g_2\left(E, \delta_U\right)\right] \end{cases}
\end{aligned}
$$