WKB analysis via topological recursion for hypergeometric differential equations

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Purpose

Theorem (Iwaki - Koike - T.)

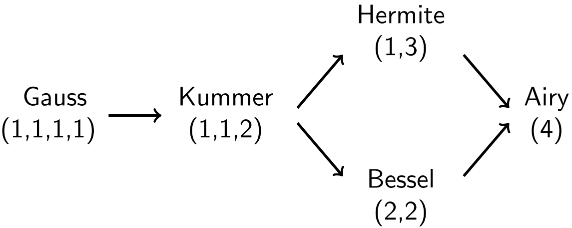
Voros coefficients in the exact WKB analysis are expressed as the difference values of the free energies for spectral curves associated with the confluent family of the Gauss hypergeometric differential equation.

Purpose of this talk

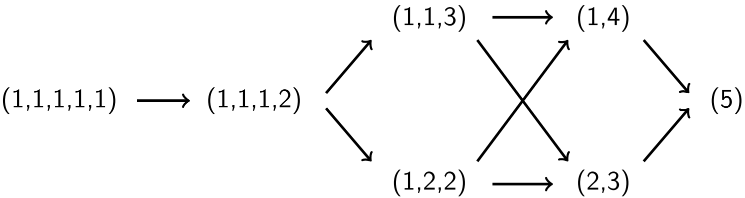
We will generalize the above result to the case of the hypergeometric differential equations associated with 2-dimensional degenerate Garnier systems.

The confluence diagram

- The family of the Gauss hypergeometric differential equations

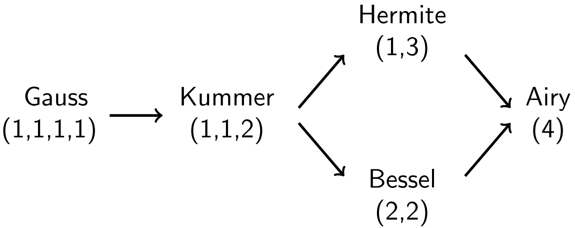


- The family of the hypergeometric differential systems associated with the 2-dimensional Garnier system ([OK])

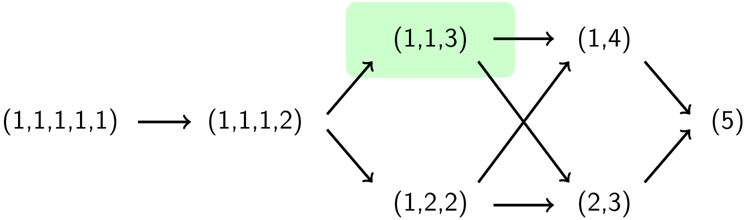


The confluence diagram

- The family of the Gauss hypergeometric differential equations



- The family of the hypergeometric differential systems associated with the 2-dimensional Garnier system ([OK])



(1,1,3) equation

The (1,1,3) hypergeometric system ($\hbar > 0$ is a small parameter) :

$$\left\{2{x_2}^2\hbar^3\frac{\partial^3}{\partial {x_1}^3} + (2x_1 - 3{x_2}^2)\hbar^2\frac{\partial^2}{\partial {x_1}^2} - (2x_1 - {x_2}^2 - 2\tilde{\lambda}_0 - 2\tilde{\lambda}_1 - 4\hbar)\hbar\frac{\partial}{\partial x_1} - 2\tilde{\lambda}_0 - 2\hbar\right\}\psi = 0, \tag{1}$$

$$\left\{x_2\hbar\frac{\partial^2}{\partial {x_1}^2} - x_2\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right\}\psi = 0,$$

where $\tilde{\lambda}_0 = \lambda_0 - \nu_0\hbar$ and $\tilde{\lambda}_1 = \lambda_1 - \nu_1\hbar$. In what follows, setting $x_1 = x$ and $x_2 = t$ (fixed), we consider

$$\left\{2t^2\hbar^3\frac{d^3}{dx^3} + (2x - 3t^2)\hbar^2\frac{d^2}{dx^2} - (2x - t^2 - 2\tilde{\lambda}_0 - 2\tilde{\lambda}_1 - 4\hbar)\hbar\frac{d}{dx} - 2\tilde{\lambda}_0 - 2\hbar\right\}\psi = 0. \tag{2}$$

1. Exact WKB analysis
2. Topological recursion
3. Main results

Exact WKB analysis, I

Let us consider the following differential equation with a small parameter $\hbar$

$$P\left(x, \hbar\frac{d}{dx}\right)\psi = \left[p_0(x)\hbar^3\frac{d^3}{dx^3} + p_1(x)\hbar^2\frac{d^2}{dx^2} + p_2(x)\hbar\frac{d}{dx} + p_3(x)\right]\psi = 0 \tag{3}$$

and its WKB solutions

$$\psi(x,\hbar) = \exp\left(\int^x S(x,\hbar)\,dx\right), \tag{4}$$

where

$$S(x,\hbar) = \hbar^{-1}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \cdots = \sum_{j \geqq -1} \hbar^j S_j(x) \tag{5}$$

is a solution of

$$\begin{aligned} &p_0(x)\hbar^3\left(\frac{d^2}{dx^2}S(x,\hbar) + 3S(x,\hbar)\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^3\right) \\ &\quad + p_1(x)\hbar^2\left(\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^2\right) + \hbar p_2(x)S(x,\hbar) + p_3(x) = 0. \end{aligned} \tag{6}$$

Exact WKB analysis, II

By substituting (5) into (6) and comparing like powers of both sides with respect to $\hbar$, we obtain

$$p_0(x)S_{-1}^3 + p_1(x)S_{-1}^2 + p_2(x)S_{-1} + p_3(x) = 0 \tag{7}$$

and

$$\begin{aligned}\left(3p_0(x){S_{-1}}^2 + 2p_1(x)S_{-1} + p_2(x)\right)S_{m+1} + \sum_{\substack{i+j+k=m-1\\ i,j,k\geq 0}} S_iS_jS_k + 3\sum_{j=0}^{m-1} S_{m-j-1}S_j \\ +3p_0(x)S_m\frac{dS_{-1}}{dx} + 3p_0(x)S_{-1}\frac{dS_m}{dx} + p_0(x)\frac{d^2S_{m-1}}{dx^2} + p_1(x)\sum_{j=0}^{m} S_{m-j}S_j \\ +p_1(x)\frac{dS_m}{dx} = 0 \quad (m\geq -1).\end{aligned} \tag{8}$$

Eq. (7) has three solutions, and once we fix one of them, we can determine S_m for $m \geq 0$ uniquely and recursively by (8).

Voros coefficients

Then, the Voros coefficient is defined by

$$
\begin{aligned}
V \text{``}=\text{''} & \int_\gamma S(x,\hbar)dx \\
&= \int_\gamma (S(x,\hbar) - \hbar^{-1} S_{-1}(x) - S_0(x))dx = \sum_{m=1}^{\infty} \hbar^m \int_\gamma S_m(x)dx,
\end{aligned} \tag{9}
$$

where γ is a path from a singular point to a singular point.

Remark: The Voros coefficient is an important ingredient to describe the global behavior of Borel resummed WKB solutions (4).

1. Exact WKB analysis
2. Topological recursion
3. Main results

Spectral curve

$$\Big[2t^2\hbar^3\frac{d^3}{dx^3} + (2x - 3t^2)\hbar^2\frac{d^2}{dx^2} - \{2x - t^2 - 2(\lambda_0 - \nu_0\hbar) - 2(\lambda_1 - \nu_1\hbar) - 4\hbar\}\hbar\frac{d}{dx}$$

$$-2\{(\lambda_0 - \nu_0\hbar) + \hbar\}\Big]\psi = 0. \quad : \text{ The (1,1,3) equation}$$

Spectral curve

$$\left[2t^2\hbar^3\frac{d^3}{dx^3} + (2x-3t^2)\hbar^2\frac{d^2}{dx^2} - \{2x - t^2 - 2(\lambda_0-\nu_0\hbar) - 2(\lambda_1-\nu_1\hbar) - 4\hbar\}\hbar\frac{d}{dx} - 2\{(\lambda_0-\nu_0\hbar)+\hbar\}\right]\psi = 0.$$

: The (1,1,3) equation

$$\Big\downarrow \ \hbar\frac{d}{dx} \to y,\ \hbar \to 0$$

$$P(x,y) = 2t^2y^3 + (2x-3t^2)y^2 - (2x-t^2-2\lambda_0-2\lambda_1)y - 2\lambda_0 = 0. \quad (10)$$

Spectral curve

$$\left[2t^2\hbar^3\frac{d^3}{dx^3}+(2x-3t^2)\hbar^2\frac{d^2}{dx^2}-\{2x-t^2-2(\lambda_0-\nu_0\hbar)-2(\lambda_1-\nu_1\hbar)-4\hbar\}\hbar\frac{d}{dx}\right.$$
$$\left.-2\{(\lambda_0-\nu_0\hbar)+\hbar\}\right]\psi=0. \quad : \text{ The (1,1,3) equation}$$

Let us consider the following algebraic curve

$$P(x,y)=2t^2y^3+(2x-3t^2)y^2-(2x-t^2-2\lambda_0-2\lambda_1)y-2\lambda_0=0. \quad (10)$$

For $z\in\mathbb{P}^1$ we choose

$$\begin{cases} x=x(z)=\dfrac{-2t^2z^3+3t^2z^2-(t^2+2\lambda_0+2\lambda_1)z+2\lambda_0}{2z(z-1)}, \\ y=y(z)=z. \end{cases} \quad (11)$$

We call a pair $(x(z),y(z))$ a spectral curve.

Topological recursion (cf. [EO1])

Let $(x(z), y(z))$ be a spectral curve. We first define

$$W_{0,1}(z) = y(z)\frac{dx}{dz}(z)dz, \quad W_{0,2}(z_1, z_2) = B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}.$$

For $g \geqq 0$, $n \geqq 0$ and $2g - 2 + n \geqq 0$, we construct meromorphic differentials $W_{g,n}(z_1, \ldots, z_n)$ on $(\mathbb{P}^1)^n$ by the following recursive formulas.

$$W_{g,n+1}(z_0, z_1, \ldots, z_n) = \sum_{a\,:\ \text{branch point}} \operatorname*{Res}_{z=a} \frac{\left(\frac{1}{z_0 - z}\right) dz_0}{(y(z) - y(\bar{z}))\, dx(z)} \times \left\{ W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1,2,\ldots,n\}}}' W_{g_1,1+|I|}(z, z_I) W_{g_2,1+|J|}(\bar{z}, z_J) \right\}. \tag{12}$$

- Branch points are zeros of $dx(z)$ (assume that all branch points are simple);
- $\bar{z}$ is a local conjugate point of z near a branch point (i.e. $x(\bar{z}) = x(z)$);
- $z_I = (z_{i_1}, \ldots, z_{i_r})$ for $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$.

Free energy (cf. [CEO])

We define $F_g = W_{g,0}$, called free energies, by the following ([EO1], [CEO]):

$$F_g = \frac{1}{2-2g} \sum_{a\,:\,\text{branch point}} \operatorname*{Res}_{z=a} \Phi(z) W_{g,1}(z) \quad (g \geqq 2), \tag{13}$$

where $\Phi(z)$ is any function satisfying $\frac{d\Phi}{dz}(z) = y(z)\,\frac{dx}{dz}(z)$.

Remark: The free energies F_0 and F_1 for $g = 0$ and 1 are also defined, but in a different manner.

Variational formula (cf. [EO2])

From the variational formula, $W_{g,n}(z_1, \ldots, z_n)$ and F_g satisfy

$$\frac{\partial W_{g,n}}{\partial \lambda_i}(z_1, \ldots, z_n) = \int_{\zeta \in \gamma_i} W_{g,n+1}(z_1, \ldots, z_n, \zeta) \quad (2g + n \geq 2), \tag{14}$$

$$\frac{\partial F_g}{\partial \lambda_i} = \int_{\zeta \in \gamma_i} W_{g,1}(\zeta) \quad (g \geq 1), \tag{15}$$

$$\frac{\partial F_g}{\partial t} = - \operatorname*{Res}_{z=\infty} (z^2 - z)\, W_{g,1}(z) \quad (g \geq 1), \tag{16}$$

where γ_0 is a path from ∞ to 0 and γ_1 is a path from ∞ to 1.

Theorem 1 (cf. [BE])

For $P(x,y) = 2t^2y^3 + (2x - 3t^2)y^2 - (2x - t^2 - 2\lambda_0 - 2\lambda_1)y - 2\lambda_0 = 0$, we define

$$\begin{aligned}\psi(x,\hbar) = \exp\Bigg[\int^z \Bigg\{\hbar^{-1} W_{0,1}(z) + \frac{1}{2!}\frac{d}{dz}\int_D\int_D\left(W_{0,2}(z_1,z_2) - \frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2}\right) \\ + \sum_{m=1}^{\infty}\hbar^m\Bigg\{\sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 1}} \frac{1}{n!}\frac{d}{dz}\int_D\cdots\int_D W_{g,n}(z_1,\dots,z_n)\Bigg\}\Bigg\}\Bigg]\Bigg|_{z=z(x)},\end{aligned} \tag{17}$$

where $z = z(x)$ is an inverse function of $x = x(z)$ and

$$\int_D = \nu_0\int_0^z + \nu_1\int_1^z + (1-\nu_0-\nu_1)\int_\infty^z .$$

Then, $\psi(x,\hbar)$ is a WKB solution of

$$\left\{2t^2\hbar^3\frac{d^3}{dx^3} + (2x-3t^2)\hbar^2\frac{d^2}{dx^2} - (2x - t^2 - 2\tilde{\lambda}_0 - 2\tilde{\lambda}_1 - 4\hbar)\hbar\frac{d}{dx} - 2\tilde{\lambda}_0 - 2\hbar\right\}\psi = 0, \tag{18}$$

where $\tilde{\lambda}_0 = \lambda_0 - \nu_0\hbar$ and $\tilde{\lambda}_1 = \lambda_1 - \nu_1\hbar$.

1. Exact WKB analysis
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Theorem 2 (for the quantum (1,1,3) curve)

Let $F_g(\lambda_0, \lambda_1, t)$ be free energies for the (1,1,3) curve and

$$F(\lambda_0, \lambda_1, t, \hbar) = \sum_{g=0}^{\infty} F_g(\lambda_0, \lambda_1, t)\hbar^{2g-2}$$

be the generating function of $F_g(\lambda_0, \lambda_1, t)$. Then, we obtain

$$V^{(\infty,0)} = F\left(\tilde{\lambda}_0+\hbar, \tilde{\lambda}_1, t; \hbar\right) - F\left(\tilde{\lambda}_0, \tilde{\lambda}_1, t; \hbar\right) - \frac{\partial F_0}{\partial \lambda_0}\hbar^{-1} + \nu_1 \frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_1} + \frac{2\nu_0 - 1}{2}\frac{\partial^2 F_0}{\partial {\lambda_0}^2},$$

$$V^{(\infty,1)} = F\left(\tilde{\lambda}_0, \tilde{\lambda}_1+\hbar, t; \hbar\right) - F\left(\tilde{\lambda}_0, \tilde{\lambda}_1, t; \hbar\right) - \frac{\partial F_0}{\partial \lambda_1}\hbar^{-1} + \nu_0 \frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_1} + \frac{2\nu_1 - 1}{2}\frac{\partial^2 F_0}{\partial {\lambda_1}^2},$$

where $V^{(\infty,i)}(\lambda_0, \lambda_1, t, \nu_0, \nu_1, \hbar)$ ($i = 0, 1$) is the Voros coefficient for the (1,1,3) hypergeometric differential equation whose path is from ∞_2 to ∞_i.

Here $x(0) = \infty_0$, $x(1) = \infty_1$, $x(\infty) = \infty_2$.

$$\left(x(z) = \frac{-2t^2z^3 + 3t^2z^2 - (t^2 + 2\lambda_0 + 2\lambda_1)z + 2\lambda_0}{2z(z-1)} \right)$$

Sketch of Proof of Theorem 2, I

From Theorem 1, we can express $\psi(x,\hbar)$ in terms of $W_{g,n}$ as follows:

$$\psi(x,\hbar) = \exp\left[\hbar^{-1}\int^{z} W_{0,1}(z) + \frac{1}{2!}\int_D\int_D \frac{dz_1\,dz_2}{(z_1z_2-1)^2} + \sum_{m=1}^{\infty}\hbar^m\left\{\sum_{\substack{2g+n-2=m\\ g\geq 0,\, n\geq 1}}\frac{1}{n!}\int_D\cdots\int_D W_{g,n}(z_1,\ldots,z_n)\right\}\right]\Bigg|_{z=z(x)}.$$

$$V^{(\infty,0)} = \int_{\infty_2}^{\infty_0}\left(S(x,\hbar)-\hbar^{-1}S_{-1}(x)-S_0(x)\right)dx$$

$$= \sum_{m=1}^{\infty}\hbar^m\int_{\infty}^{0}\left\{\sum_{2g+n-2=m}\frac{1}{n!}\frac{d}{dz}\int_{D(z)}\cdots\int_{D(z)} W_{g,n}(z_1,\ldots,z_n)\right\}dz$$

$$= \sum_{m=1}^{\infty}\hbar^m\sum_{\substack{2g-2+n=m\\ g\geq 0,\, n\geq 1}}\frac{1}{n!}\left\{\left(-\nu_0\int_{\gamma_0}-\nu_1\int_{\gamma_1}\right)^n-\left((1-\nu_0)\int_{\gamma_0}-\nu_1\int_{\gamma_1}\right)^n\right\}W_{g,n},$$

$$\left(\int_{D(z)} = \nu_0\int_0^z+\nu_1\int_1^z+(1-\nu_0-\nu_1)\int_{\infty}^{z},\quad \int_{\gamma_0}=\int_{\infty}^{0},\quad \int_{\gamma_1}=\int_{\infty}^{1}\right)$$

where we use the notation $\left(\int_\gamma\right)^n W_{g,n} = \int_{\zeta_1\in\gamma}\cdots\int_{\zeta_n\in\gamma} W_{g,n}(\zeta_1,\ldots,\zeta_n)$.

Sketch of Proof of Theorem 2, II

$$V^{(\infty,0)} = \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 1}} \left[\sum_{k=1}^{n} \frac{\{(-\nu_0)^k - (1-\nu_0)^k\}(-\nu_1)^{n-k}}{k!(n-k)!} \left\{\int_{\gamma_0}\right\}^k \left\{\int_{\gamma_1}\right\}^{n-k} W_{g,n} \right]$$

$$= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 1}} \left[\sum_{k=1}^{n} \frac{\{(-\nu_0)^k - (1-\nu_0)^k\}(-\nu_1)^{n-k}}{k!(n-k)!} \frac{\partial^n F_g}{\partial {\lambda_0}^k \partial {\lambda_1}^{n-k}} \right]$$

$$= \sum_{n\geq 1} \left[\sum_{k=1}^{n} \frac{\{(-\nu_0)^k - (1-\nu_0)^k\}(-\nu_1)^{n-k}}{k!(n-k)!} \frac{\partial^n}{\partial {\lambda_0}^k \partial {\lambda_1}^{n-k}} \left\{ \sum_{g\geq 0} \hbar^{2g-2} F_g(\lambda_0, \lambda_1) \right\} \hbar^n \right]$$

$$- \frac{\partial F_0}{\partial \lambda_0}\hbar^{-1} + \nu_1 \frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_1} + \frac{2\nu_0 - 1}{2} \frac{\partial^2 F_0}{\partial {\lambda_0}^2}$$

$$= F\left(\tilde{\lambda}_0 + \hbar, \tilde{\lambda}_1, t; \hbar\right) - F\left(\tilde{\lambda}_0, \tilde{\lambda}_1, t; \hbar\right) - \frac{\partial F_0}{\partial \lambda_0}\hbar^{-1} + \nu_1 \frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_1} + \frac{2\nu_0 - 1}{2} \frac{\partial^2 F_0}{\partial {\lambda_0}^2},$$

where we use the variational formula

$$\frac{\partial W_{g,n}}{\partial \lambda_i}(z_1, \ldots, z_n) = \int_{\zeta\in\gamma_i} W_{g,n+1}(z_1, \ldots, z_n, \zeta) \quad (2g+n \geq 2), \tag{19}$$

$$\frac{\partial F_g}{\partial \lambda_i} = \int_{\zeta\in\gamma_i} W_{g,1}(\zeta) \quad (g \geq 1). \tag{20}$$

Explicit form of free energies (for the (1,1,3) curve), I

From Theorem 2 we obtain

$$F(\lambda_0 + \hbar, \lambda_1, t; \hbar) - 2F(\lambda_0, \lambda_1, t; \hbar) + F(\lambda_0 - \hbar, \lambda_1, t; \hbar) = \frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \tag{21}$$

$$F(\lambda_0, \lambda_1 + \hbar, t; \hbar) - 2F(\lambda_0, \lambda_1, t; \hbar) + F(\lambda_0, \lambda_1 - \hbar, t; \hbar) = \frac{\partial^2 F_0}{\partial {\lambda_1}^2}. \tag{22}$$

(21) is rewritten as

$$\left\{ e^{\hbar \frac{\partial}{\partial \lambda_0}} - 2 + e^{-\hbar \frac{\partial}{\partial \lambda_0}} \right\} F(\underline{\lambda}, t, \hbar) = e^{-\hbar \frac{\partial}{\partial \lambda_0}} \left(e^{\hbar \frac{\partial}{\partial \lambda_0}} - 1 \right)^2 F(\underline{\lambda}, t, \hbar) = \frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

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$$\frac{e^w}{(e^w-1)^2}=\frac{1}{w^2}-\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g-2)!}w^{2g-2} \quad \downarrow$$

$$F=F_0(\underline{\lambda},t)\hbar^{-2}+F_1(\underline{\lambda},t)+\sum_{g=2}^{\infty}\left\{\frac{B_{2g}}{2g(2g-2)}{\lambda_0}^{2-2g}+ \qquad G(\lambda_1,t) \qquad \right\}\hbar^{2g-2}.$$

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$$\frac{B_{2g}}{2g(2g-2)}{\lambda_1}^{2-2g}+G(t)$$

Explicit form of free energies (for the (1,1,3) curve), II

From variational formula we obtain

$$\frac{\partial F_g}{\partial t} = -\operatorname*{Res}_{z=\infty} (z^2 - z)\, W_{g,1}(z) \quad (g \geq 1). \tag{23}$$

On the other hand, by substituting $\nu_0 = \nu_1 = 0$ in (17), we get

$$\sum_{m=-1}^{\infty} \hbar^m \int^{x(z)} S_m dx = \sum_{m=-1}^{\infty} \hbar^m \left\{ \sum_{\substack{2g+n-2=m \\ g \geq 0,\, n \geq 1}} \frac{1}{n!} \int_\infty^z \cdots \int_\infty^z W_{g,n}(z_1, \ldots, z_n) \right\}, \tag{24}$$

$$\operatorname*{Res}_{z=\infty}(z^2 - z) \sum_{m=-1}^{\infty} \hbar^m S_m(x(z)) dx(z)$$

$$= \operatorname*{Res}_{z=\infty}(z^2 - z) \sum_{g \geq 0} \hbar^{2g-1} W_{g,1}(z) + \operatorname*{Res}_{z=\infty}(z^2 - z) \int_\infty^z W_{0,2}(z, z_2)$$

$$+ \operatorname*{Res}_{z=\infty}(z^2 - z) \sum_{\substack{g \geq 0,\, n \geq 2 \\ (g,n) \neq (0,2)}} \frac{\hbar^{2g-2+n}}{(n-1)!} \int_\infty^z \cdots \int_\infty^z W_{g,n}(z, z_2, \ldots, z_n). \tag{25}$$

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$$\sum_{m=-1}^{\infty} \hbar^m \int^{x(z)} S_m dx = \sum_{m=-1}^{\infty} \hbar^m \left\{ \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 1}} \frac{1}{n!} \int_{\infty}^{z} \cdots \int_{\infty}^{z} W_{g,n}(z_1, \ldots, z_n) \right\}, \tag{24}$$

$$\operatorname*{Res}_{z=\infty}(z^2 - z) \sum_{m=-1}^{\infty} \hbar^m S_m(x(z)) dx(z)$$

$$= \operatorname*{Res}_{z=\infty}(z^2 - z) \sum_{g\geq 0} \hbar^{2g-1} W_{g,1}(z) + \operatorname*{Res}_{z=\infty}(z^2 - z) \int_{\infty}^{z} W_{0,2}(z, z_2)$$

$$+ \operatorname*{Res}_{z=\infty}(z^2 - z) \sum_{\substack{g\geq 0,\, n\geq 2 \\ (g,n)\neq(0,2)}} \frac{\hbar^{2g-2+n}}{(n-1)!} \int_{\infty}^{z} \cdots \int_{\infty}^{z} W_{g,n}(z, z_2, \ldots, z_n). \tag{25}$$

$$\longrightarrow \frac{\partial F_g}{\partial t} = \frac{\partial G}{\partial t} = 0 \quad (g \geq 1). \qquad \longrightarrow G(t) = 0.$$

Explicit form of the Voros coeff. (for the (1,1,3) equation)

Using the explicit form of the free energy

$$F_g(\lambda_0, \lambda_1) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{{\lambda_0}^{2g-2}} + \frac{1}{{\lambda_1}^{2g-2}} \right\} \quad (g \geq 2) \tag{26}$$

and Theorem 2, we get the explicit forms of the Voros coefficients of the (1,1,3) hypergeometric differential equation.

Explicit forms of the Voros coefficients (for the (1,1,3) equation)

$$V^{(\infty,0)}(\underline{\lambda}, t, \underline{\nu}, \hbar) = \sum_{m=2}^{\infty} \frac{B_m(\nu_0)}{m(m-1)} \left(\frac{\hbar}{\lambda_0} \right)^{m-1}, \tag{27}$$

$$V^{(\infty,1)}(\underline{\lambda}, t, \underline{\nu}, \hbar) = \sum_{m=2}^{\infty} \frac{B_m(\nu_1)}{m(m-1)} \left(\frac{\hbar}{\lambda_1} \right)^{m-1}, \tag{28}$$

where $B_m(X)$ designates the m-th Bernoulli polynomial defined by

$$\frac{we^{Xw}}{e^w - 1} = \sum_{m=0}^{\infty} \frac{B_m(X)}{m!} w^m.$$

Summary

For the (1,4) HG equation, the (2,3) HG equation and the (1,1,3) equation, we obtain the following results:

- Voros coefficients are expressed as the difference values of the generating function of the free energies with respect to parameters.
  $\longrightarrow$ It means that the Voros coefficients are controlled by the free energy, in other words, the free energy is more essential quantity.
- As its applications, we get the explicit forms of the Voros coefficients and free energies.

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Thank you for your attention !

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