Propagation of singularities for Schrödinger equations on manifolds with ends

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Himeji conference on Partial Differential Equations
March 2–4, 2022

Contents

Homogeneous wavefront sets (HWF)

Definition 1

For $u \in L^2(\mathbb{R}^n)$, we define a homogeneous wavefront set $\mathrm{HWF}(u) \subset T^*\mathbb{R}^n \setminus \{(0,0)\}$ as follows: a point $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{(0,0)\}$ is *not* in $\mathrm{HWF}(u)$ if there exists a symbol $a \in C_c^\infty(T^*\mathbb{R}^n)$ such that

- $a = 1$ near (x_0, ξ_0),
- $\|a^{\mathrm{w}}(\hbar x, \hbar D)u\|_{L^2} = O(\hbar^\infty)$.

Here $a^{\mathrm{w}}(\hbar x, \hbar D)$ is defined as

$$a^{\mathrm{w}}(\hbar x, \hbar D)u(x) := \frac{1}{(2\pi)^n} \int_{T^*\mathbb{R}^n} a\left(\frac{\hbar x + \hbar y}{2}, \hbar \xi\right) e^{i\xi\cdot(x-y)} u(y)\, dy d\xi.$$

Theorem 2 (Nakamura (2005))

Let H *be a Hamiltonian of the form*

$$H := -\frac{1}{2}\sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(x)\partial_{x_k}) + V(x)$$

with

- $|\partial_x^\alpha(a_{jk}(x)-\delta_{jk})| \le C_\alpha \langle x\rangle^{-\mu-|\alpha|}$ *(*$\exists\mu>0$*),*
- $|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{\nu-|\alpha|}$ *(*$\exists\nu<2$*).*

Let $(x(t),\xi(t))$ *be a nontrapping classical orbit with respect to the Hamiltonian* $h_0 := \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k/2$ *and let* $\xi_\infty := \lim_{t\to\infty}\xi(t)$*. Then, for any* $t_0>0$ *and* $u\in L^2(\mathbb{R}^n)$*,*

$$(x(0),\xi(0))\in \mathrm{WF}(u) \Longrightarrow (t_0\xi_\infty,\xi_\infty)\in \mathrm{HWF}(e^{-it_0H}u).$$

HWF on manifolds?

On manifolds: "$\hbar x$"??? Instead of $\hbar x$, we want to consider polar coordinates (r, θ) and replace $\hbar x$ to $(\hbar r, \theta)$.
$\Longrightarrow$ We want to describe the regularity such as

$$a^{\mathrm{w}}(\hbar r, \theta, \hbar D_r, \hbar D_\theta)u = O_{L^2}(\hbar^\infty).$$

$\Longleftrightarrow$ radially homogeneous wavefront sets (Ito-Nakamura (Amer. J. Math., 2009)).

Remark

- The (complement of) HWF is described as
$$a^{\mathrm{w}}(\hbar r, \theta, \hbar D_r, \hbar^2 D_\theta)u = O_{L^2}(\hbar^\infty).$$
- For $x \neq 0$, $(x, \xi) \in \mathrm{WF}^{\mathrm{rh}}(u) \Longrightarrow (x, (\xi \cdot \hat{x})\hat{x}) \in \mathrm{HWF}(u)$ where $\hat{x} := x/|x|$.

Manifolds with ends

Let M be an n-dimensional non-compact manifold.

Assumption 1

There exist

- an open subset E of M,
- an $(n-1)$-dimensional compact manifold S and
- a diffeomorphism $\Psi : E \to \mathbb{R}_+ \times S$

such that the set $M \setminus \Psi^{-1}((1, \infty))$ is a compact subset of M.
Here $\mathbb{R}_+ := (0, \infty)$.

The set E is called the end of M.

Intrinsic L^2 space

Let $C_c^\infty(M;\Omega^{1/2})$ be the space of compactly supported smooth half-densities on M. Then we introduce an inner product on $C_c^\infty(M;\Omega^{1/2})$ defined as

$$\langle u, v\rangle := \int_M \tilde{u}(x)\overline{\tilde{v}(x)}\,dx$$

where $u = \tilde{u}|dx|^{1/2}$ and $v = \tilde{v}|dx|^{1/2}$ locally. The intrinsic L^2 space $L^2(M;\Omega^{1/2})$ is defined as the completion of $C_c^\infty(M;\Omega^{1/2})$ by the inner product $\langle\cdot,\cdot\rangle$.

Hamiltonian

We consider a Hamiltonian

$$H = -\frac{1}{2}\triangle_g + V(x)$$

where

- $\triangle_g$ is the associated Laplacian with respect to a fixed metric g on M,
- $V \in C^\infty(M;\mathbb{R})$ is a potential function.

$\triangle_g$ acts on half-density $u = \tilde{u}|\mathrm{vol}_g(x)|^{1/2}$ as

$$\triangle_g(\tilde{u}|\mathrm{vol}_g(x)|^{1/2}) := (\triangle_g\tilde{u})|\mathrm{vol}_g(x)|^{1/2}.$$

Metric

Assumption 2

- The metric g is the form
$$\Psi_* g(r, \theta, dr, d\theta) = c(r, \theta)^2 dr^2 + h(r, \theta, d\theta)$$
where $c(r, \theta) > 0$ and $h(r, \theta, d\theta)$ is a r-dependent metric on S.
- $C^{-1} f(r)^2 h(1, \theta, d\theta) \le h(r, \theta, d\theta) \le C f(r)^2 h(1, \theta, d\theta)$ for some constant $C > 0$ and a smooth function $f : \mathbb{R} \to \mathbb{R}_+$ with
$$c_0 r^{-1} \le f'(r)/f(r) \le C \quad (r \ge 1)$$
for some $c_0 > 1/2$.

Metric

Assumption 2 (continued)

- For all multiindices $\alpha = (\alpha_0, \alpha') \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{n-1}$, the estimates

$$|\partial_r^{\alpha_0} \partial_\theta^{\alpha'} (c(r,\theta) - 1)| \leq C_\alpha r^{-1-\mu},$$
$$\left| \sum_{j,k=1}^{n-1} \partial_r^{\alpha_0} \partial_\theta^{\alpha'} h_{jk}(r,\theta) w_j w_k \right| \leq C_\alpha h(r,\theta,w) \quad (\forall w \in T_\theta S),$$
$$|\partial_r^{\alpha_0} \partial_\theta^{\alpha'} V(r,\theta)| \leq C_\alpha.$$

Classical free Hamiltonian

- $(\rho, \eta) \in T^*_{(r,\theta)}M$: dual variable of (r, θ).

The classical free Hamiltonian is

$$h_0(r, \theta, \rho, \eta) = \frac{1}{2}\left(c(r, \theta)^{-2}\rho^2 + h^*(r, \theta, \eta)\right).$$

Here $h^*(r, \theta, \eta)$ is the dual metric

$$h^*(r, \theta, \eta) := \sum_{j,k=1}^{n-1} h^{jk}(r, \theta)\eta_j\eta_k$$

where $h^{jk}(r, \theta)$ is the inverse matrix of $(h_{jk}(r, \theta))_{j,k=1}^{n-1}$ defined as

$$h(r, \theta, d\theta) = \sum_{j,k=1}^{n-1} h_{jk}(r, \theta)d\theta_j d\theta_k.$$

Classical analogue of Mourre estimate

Assumption 3

$$\{f\rho, h_0\} \geq 2f'(r)(h_0 - Cr^{-1-\mu})$$

holds for all $(r, \theta, \rho, \eta) \in T^*E \cap \{r \geq 1\}$.

Theorem 3

Let $(r(t), \theta(t), \rho(t), \eta(t))$ be a nontrapping ($r(t) \to \infty$ as $t \to \infty$) classical orbit with respect to the Hamiltonian h_0. Then, under Assumption 1–3,

$$\exists(\rho_\infty, \theta_\infty, \eta_\infty) := \lim_{t\to\infty}(\rho(t), \theta(t), \eta(t)) \in \mathbb{R}_+ \times T^*S.$$

Radially homogeneous wavefront sets

Definition 4

For $u \in L^2(M; \Omega^{1/2})$, we define a radially homogeneous wavefront set $\mathrm{WF}^{\mathrm{rh}}(u) \subset T^*E$ as follows: a point $(x_0, \xi_0) \in T^*E$ does *not* in $\mathrm{WF}^{\mathrm{rh}}(u)$ if there exist $a \in C_c^\infty(T^*\mathbb{R}^n)$, $\chi \in C^\infty(M)$ and a polar coordinate function
$\varphi : U\,(\subset E) \to V = \mathbb{R}_+ \times V'\,(\subset \mathbb{R}_+ \times S)$ near x_0 such that

- $\operatorname{supp} a \subset \tilde{\varphi}(T^*E)$ and $a = 1$ near $\tilde{\varphi}(x_0, \xi_0)$,
- $\chi(r, \theta) = \exists \chi_{\mathrm{ang}}(\theta)$ for $r \gg 1$, $\operatorname{supp} \chi \subset U$, and $\chi = 1$ near $\Psi^{-1}([R, \infty) \times \{\theta_\infty\})$ for some $R > 0$,
- $\|\chi \varphi^* a^{\mathrm{w}}(\hbar r, \theta, \hbar D_r, \hbar D_\theta) \varphi_*(\chi u)\|_{L^2} = O(\hbar^\infty)$.

Main theorem

Theorem 5 (F, arXiv:2201.09466 [math.AP])

Suppose Assumption 1–3. Let $u \in L^2(M; \Omega^{1/2})$ *and* $(x(t), \xi(t)) = \Psi^{-1}(r(t), \theta(t), \rho(t), \eta(t))$ *is a nontrapping classical orbit. Then, for any* $t_0 > 0$, $(x(0), \xi(0)) \in \mathrm{WF}(u)$ *implies* $\Psi^{-1}(\rho_\infty t_0, \theta_\infty, \rho_\infty, \eta_\infty) \in \mathrm{WF}^{\mathrm{rh}}(e^{-it_0 H} u)$.

Symbol class

Definition 6

For $m \in \mathbb{R}$, we define a symbol class $S^m_{\mathrm{cyl}}(T^*M) \subset C^\infty(T^*M)$ as follows:

- For polar coordinates (r, θ), the estimate

$$|\partial_r^{\alpha_0} \partial_\theta^{\alpha'} \partial_\rho^{\beta_0} \partial_\eta^{\beta'} a(r, \theta, \rho, \eta)| \leq C_{\alpha\beta}(1 + |\rho| + |\eta|)^{m-|\beta|}$$

  holds for all multiindices $\alpha = (\alpha_0, \alpha')$, $\beta = (\beta_0, \beta') \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{n-1}$.
- The conditions of usual Kohn-Nirenberg symbols are satisfied on $M \setminus E$.

Quantization

We fix

- a finite atlas $\{\varphi_\iota : U_\iota \to V_\iota\}_{\iota \in I}$,
- a partition of unity $\{\kappa_\iota \in C^\infty(M)\}_{\iota \in I}$ subordinate to the atlas,
- a family of functions $\{\chi_\iota \in C^\infty(M)\}_{\iota \in I}$ such that $\operatorname{supp} \chi_\iota \subset U_\iota$ and $\chi_\iota = 1$ near $\operatorname{supp} \kappa_\iota$,

and define

$$\mathrm{Op}_\hbar(a)u := \sum_{\iota \in I} \chi_\iota \varphi_\iota^* (\tilde{\varphi}_{\iota *} a)^{\mathrm{w}}(x, \hbar D) \varphi_* (\chi_\iota u)$$

for $a \in S^m_{\mathrm{cyl}}(T^*M)$ and $u \in C_c^\infty(M; \Omega^{1/2})$.

Quantization

The index set I is decomposed into $I = I_K \cup I_\infty$:

- $\{U_\iota\}_{\iota \in I_K}$ covers $M \setminus E$ and $\kappa_\iota, \chi_\iota \in C_c^\infty(U_\iota)$, and
- $\{U_\iota\}_{\iota \in I_\infty}$ is a family of polar coordinates on E and $\kappa_\iota, \chi_\iota \in C^\infty(U_\iota)$ depend only on θ near infinity.

Basic properties as in the usual pseudodifferential operators hold:

- **Calderón-Vaillancourt theorem:** $\|\operatorname{Op}_\hbar(a)\|_{L^2 \to L^2} \leq C \sum_{|\alpha| \leq N} |\partial^\alpha a|$ for $a \in S^0_{\mathrm{cyl}}(T^*M)$.
- **Sharp Gårding inequality:** $\operatorname{Re} \operatorname{Op}_\hbar(a) \geq -\operatorname{Op}_\hbar(b) + O_{L^2 \to L^2}(\hbar^\infty)$ for any $a \in S^0_{\mathrm{cyl}}(T^*M)$ with $\operatorname{Re} a \geq 0$ and some $b \in S^0_{\mathrm{cyl}}(T^*M)$ with $\operatorname{supp} b \subset \operatorname{supp} a$ modulo $O(\hbar^\infty)$.

Quantization

- **Composition:** For $a \in S^{m_1}_{\mathrm{cyl}}(T^*M)$ and $b \in S^{m_2}_{\mathrm{cyl}}(T^*M)$,

$$\mathrm{Op}_\hbar(a)\,\mathrm{Op}_\hbar(b) = \mathrm{Op}_\hbar(c) + O_{L^2\to L^2}(\hbar^\infty),$$
$$[\mathrm{Op}_\hbar(a), \mathrm{Op}_\hbar(b)] = i\hbar\,\mathrm{Op}_\hbar(c') + O_{L^2\to L^2}(\hbar^\infty)$$

for some $c = ab + O_{S^{m_1+m_2-1}_{\mathrm{cyl}}(T^*M)}(\hbar)$ and $c' = \{a, b\} + O_{S^{m_1+m_2-2}_{\mathrm{cyl}}(T^*M)}(\hbar)$ with $\mathrm{supp}\, c, \mathrm{supp}\, c' \subset \mathrm{supp}(ab)$ modulo $O(\hbar^\infty)$.

Remark: ΨDOs acting on half-densities

Remark

On Euclidean spaces.

$$a^{\mathrm{w}}(x,\hbar D)(\tilde{u}|dx|^{1/2}) := (a^{\mathrm{w}}(x,\hbar D)\tilde{u})|dx|^{1/2}.$$

On curved spaces. Let g be a general metric on $\mathbb{R}^n$ and $g(x) := \det(g_{jk}(x))$. Then, noting that the natural identification $\tilde{u}|g^{1/2}dx|^{1/2} \simeq \tilde{u}$, we have

$$a^{\mathrm{w}}(x,\hbar D)(\tilde{u}|g^{1/2}dx|^{1/2}) = g^{-1/4}a^{\mathrm{w}}(x,\hbar D)(\tilde{u}g^{1/4})|g^{1/2}dx|^{1/2}.$$

Construction of symbols

We follow the argument in Ito-Nakamura (Amer. J. Math., 2009). Take $\chi \in C_c^\infty(\mathbb{R}$ with $\chi = 1$ in $[-1, 1]$ and $\chi = 0$ outside $[-2, 2]$. For $j = 0, 1, 2, \ldots$, we consider

$$\tilde{\psi}_j(t, r, \theta, \rho, \eta) := \chi\left(\frac{|r - r(t)|}{4\delta_j t}\right) \chi\left(\frac{|\theta - \theta(t)|}{\delta_j - t^{-\lambda}}\right) \times \chi\left(\frac{|\rho - \rho(t)|}{\delta_j - t^{-\lambda}}\right) \chi\left(\frac{|\eta - \eta(t)|}{\delta_j - t^{-\lambda}}\right)$$

for $t \geq T(\gg 1)$. $\delta_j > 0$ and $\lambda > 0$ are chosen appropriately.

Construction of symbols

Take a function $\alpha \in C^\infty(\mathbb{R})$ with $\alpha = 0$ in $(-\infty, T)$ and $\alpha = 1$ in $(T+1, \infty)$, and we "extend" $\tilde{\psi}_j(t, \dots)$ $(t \geq T)$ to $\psi_j(t, \dots)$ $(t \geq 0)$ by the transport equation

$$\frac{\partial \psi_j}{\partial t} + \{\psi_j, h_0\} = \alpha(t) \left(\frac{\partial \tilde{\psi}_j}{\partial t} + \{\tilde{\psi}_j, h_0\} \right),$$

$$\psi_j(T+1, r, \theta, \rho, \eta) = \tilde{\psi}_j(T+1, r, \theta, \rho, \eta).$$

Lemma 7

ψ_j belongs to $S^{-2}_{\mathrm{cyl}}(T^*M)$ and satisfies

$$\frac{\partial \psi_j}{\partial t} + \{\psi_j, h_0\} \geq 0, \quad \frac{\partial \psi_j}{\partial t} + \{\psi_j, h_0\} = O_{S^0_{\mathrm{cyl}}(T^*M)}(\langle t \rangle^{-1}).$$

Construction of symbols

We define a symbol of the form

$$\tilde{a}(\hbar; t, x, \xi) \sim t \sum_{j=1}^{\infty} c_j \hbar^j \psi_j(t, x, \xi)$$

where c_j's are positive constants, and consider

$$A_\hbar(t) := \mathrm{Op}_\hbar(\psi_0(\hbar^{-1}t))^* \mathrm{Op}_\hbar(\psi_0(\hbar^{-1}t)) + \mathrm{Op}_\hbar(\tilde{a}(\hbar; \hbar^{-1}t)).$$

we set

$$F_k(t) := \mathrm{Op}_\hbar(\psi_0(t))^* \mathrm{Op}_\hbar(\psi_0(t)) + t \sum_{j=1}^{k} c_j \hbar^j \psi_j(t, x, \xi).$$

Heisenberg derivatives

Note that $H = \hbar^{-2}(\mathrm{Op}_\hbar(h_0) + \hbar^2 V + \hbar^2 V_g/2)$ for some $V_g(r,\theta) \in S^0_{\mathrm{cyl}}(T^*M)$. We apply the sharp Gårding inequality for $F_0(t)$ and obtain:

Lemma 8

There exists a symbol $b_0(\hbar; t, x, \xi) \in S^0_{\mathrm{cyl}}(T^*M)$ *such that*

$$\frac{\partial}{\partial t} F_0(t) - i\hbar[F_0(t), H] \geq -\hbar\,\mathrm{Op}_\hbar(b_0(t)) + O_{L^2 \to L^2}(\hbar^\infty)$$

with $\operatorname{supp} b_0(t) \subset \operatorname{supp} \psi_0(t)$ *modulo* $O(\hbar^\infty)$.

Heisenberg derivative

If we take $c_1 \gg 1$, $F_1(t)$ satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t} F_1(t) - i\hbar[F_1(t), H] \\
&\geq \underbrace{-\hbar \operatorname{Op}_\hbar(b_0(t)) + c_1\hbar \operatorname{Op}_\hbar(\psi_1(t))}_{\geq -\hbar^2 \operatorname{Op}_\hbar S^0_{\mathrm{cyl}}(T^*M)} \\
&\quad + c_1 t\hbar \underbrace{\left(\frac{\partial}{\partial t} \operatorname{Op}_\hbar(\psi_1(t)) - i\hbar[\operatorname{Op}_\hbar(\psi_1(t)), H] \right)}_{\geq -\hbar \langle t \rangle^{-1} \operatorname{Op}_\hbar S^0_{\mathrm{cyl}}(T^*M)} + O_{L^2 \to L^2}(\hbar^\infty) \\
&\geq -\hbar^2 \operatorname{Op}_\hbar(b_1(t)) + O_{L^2 \to L^2}(\hbar^\infty)
\end{aligned}$$

for some $b_1(\hbar; t) \in S^0_{\mathrm{cyl}}(T^*M)$ with $\operatorname{supp} b_1(t) \subset \operatorname{supp} \psi_1(t)$ modulo $O(\hbar^\infty)$.

Heisenberg derivative

We repeat the same procedure and obtain

$$\frac{\partial}{\partial t}F_k(t) - i\hbar[F_k(t), H] \geq O_{L^2\to L^2}(\hbar^{k+1}).$$

Noting that $A_\hbar(t) - F_k(\hbar^{-1}t) = O_{L^2\to L^2}(\hbar^k)$, we obtain

$$\frac{\partial}{\partial t}A_\hbar(t) - i[A_\hbar(t), H] \geq O_{L^2\to L^2}(\hbar^k)$$

for any $k \geq 0$.

Heisenberg derivative

We have

$$\begin{aligned}
&\langle A_\hbar(0)u, u\rangle \\
&= \langle A_\hbar(t_0)e^{-it_0H}u, e^{-it_0H}u\rangle \\
&\quad - \int_0^{t_0} \left\langle \left(\frac{\partial}{\partial t}A_\hbar(t) - i[A_\hbar(t), H]\right) e^{-itH}u, e^{-itH}u \right\rangle dt \\
&\leq \langle A_\hbar(t_0)e^{-it_0H}u, e^{-it_0H}u\rangle + O(\hbar^\infty).
\end{aligned}$$

By $\langle A_\hbar(t_0)u, u\rangle = \| \operatorname{Op}_\hbar(\psi_0(0))u\|_{L^2}^2$ and $\psi_0(0) = 1$ near $(x(0), \xi(0))$, we only have to prove

$$\langle A_\hbar(t_0)e^{-it_0H}u, e^{-it_0H}u\rangle = O(\hbar^\infty).$$

Estimate of expectation value

Roughly speaking, this estimate holds since
$A_\hbar(t_0) = \mathrm{Op}_\hbar(\psi_0(\hbar^{-1}t_0))^* \mathrm{Op}_\hbar(\psi_0(\hbar^{-1}t_0)) + \cdots$ and

$$\begin{aligned}\psi_j(\hbar^{-1}t_0) &= \chi\left(\frac{|\hbar r - \hbar r(\hbar^{-1}t_0)|}{4\delta_j t_0}\right)\chi\left(\frac{|\theta - \theta(\hbar^{-1}t_0)|}{\delta_j - (\hbar^{-1}t_0)^{-\lambda}}\right) \\ &\quad \times \chi\left(\frac{|\rho - \rho(\hbar^{-1}t_0)|}{\delta_j - (\hbar^{-1}t_0)^{-\lambda}}\right)\chi\left(\frac{|\eta - \eta(\hbar^{-1}t_0)|}{\delta_j - (\hbar^{-1}t_0)^{-\lambda}}\right) \\ &\approx \chi\left(\frac{|\hbar r - \rho_\infty t_0|}{4\delta_j t_0}\right)\chi\left(\frac{|\theta - \theta_\infty|}{\delta_j}\right) \\ &\quad \times \chi\left(\frac{|\rho - \rho_\infty|}{\delta_j}\right)\chi\left(\frac{|\eta - \eta_\infty|}{\delta_j}\right)\end{aligned}$$

for $0 < \hbar \ll 1$. □