Topological Levinson's theorem with two Hilbert spaces: from finite graphs to insulators

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Introduction

Original version of Levinson's theorem

Levinson's theorem is a fundamental relation in QM originally established by N. Levinson in 1949.

Theorem

Consider the radial Schrödinger operator

$$H = -\partial_r^2 + v(r) \Big(\equiv H_{\ell=0} \Big) \quad \textit{with Dirichlet b.c.}$$

acting on $\mathcal{H} = L^2(\mathbb{R}_+)$ *with a potential* v *decaying faster than* r^{-2} *as* $r \to \infty$*. Then, one has the relation*

$$\textit{scattering phase shift} = \dim \mathcal{H}_{\mathrm{pp}}(H) + \delta,$$

where the correction term δ *is given by*

$$\delta = \begin{cases} 1/2 & \textit{if } H \textit{ has zero-energy resonance,} \\ 0 & \textit{otherwise.} \end{cases}$$

Topological version of Levinson's theorem

Topological approach has been widely popularized.

Description with the previous model cf. [Kellendonk& Richard, 2007]

Let $H_0 := -\partial_r^2$ with Dirichlet boundary condition.

1. The wave operators exist and are complete:

$$W_\pm := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-itH_0}$$

2. One constructs a C^*-subalgebra $\mathcal{E} \subset \mathcal{B}(\mathcal{H})$ satisfying

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{E} \xrightarrow{\pi} C(\mathbb{T}) \to 0 \qquad \text{(exact)},$$

   $W_- \in \mathcal{E}$ and $\pi(W_-) \in C(\mathbb{T})$ is unitary.

3. It follows from K-theory that the following index theorem holds:

$$\mathrm{Wind}(\pi(W_-)) = -\,\mathrm{Index}(W_-).$$

4. One observes that the above index theorem is equivalent to Levinson's theorem.

Scattering theory with two Hilbert spaces

In practice, free and perturbed systems may act on different spaces, i.e.

$$H_0 \curvearrowright \mathcal{H}_0, \quad H \curvearrowright \mathcal{H} \quad \text{and} \quad \mathcal{H}_0 \neq \mathcal{H}.$$

Then one may introduce an identification operator $J : \mathcal{H}_0 \to \mathcal{H}$ to construct the wave operator as

$$\boldsymbol{W}_\pm \equiv W_\pm(H, H_0; J) := \underset{t \to \pm\infty}{\text{s-lim}}\, \mathrm{e}^{itH} J \mathrm{e}^{-itH_0}$$

Motivating Question Can we still give a topological interpretation for Levinson's theorem in two Hilbert spaces setting?

Problem

C^*-algebras are defined as a subalgebra of $\mathcal{B}(\mathcal{G})$ for a Hilbert space $\mathcal{G}$. However, $\boldsymbol{W}_\pm \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$ whenever they exist.

$\longrightarrow$ There is **no** C^*-algebra $\mathcal{E}$ containing the two-space WO $\boldsymbol{W}_-$.

Let us discuss this problem with a simple example!

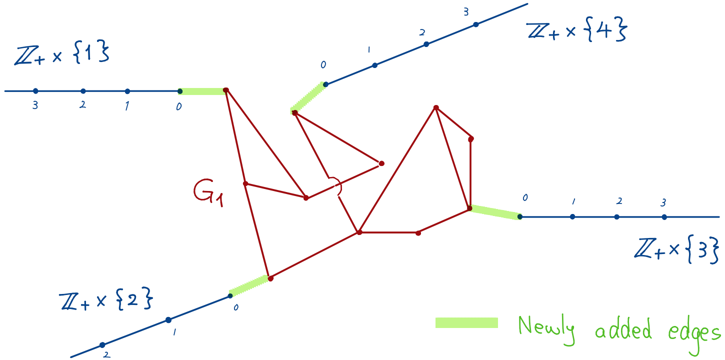
Generalized Childs' Graph Model

Childs' graph model

Childs' graph model [Childs et. al. 2011 & 2012] consists of a bundle of wires attached to a finite graph G_1.

- H_0: the Laplacian on the bundle of wires $\mathbb{Z}_+ \times \mathcal{V}$ with $\mathcal{V}$ finite,
- H: the Laplacian on the graph G obtained by attaching wires to G_1

Ex. $\mathcal{V} = \{1, 2, 3, 4\}$



Levinson's theorem for Childs' graph model

Theorem (Childs et. al. 2011 & 2012)

Set $s(\lambda) := \det(S(\lambda))$ *with* $S(\lambda) \in \mathcal{B}(\ell^2(\mathcal{V}))$ *being the scattering matrix for the pair* (H_0, H)*. Then, one has*

$$\frac{1}{2\pi i}\int_{\sigma(H_0)} s'(\lambda)s(\lambda)^*\mathrm{d}\lambda = \#\{\textit{bound states}\} + \delta - \#\{\textit{vertices of } G_1\}$$

New Question Why does $\#\{\text{vertices of } G_1\}$ appear?

Generalized Childs' graph model

The free system H_0 (wires)

Let $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ and $\mathcal{V}$ be a countable set.

- A (free) particle in a bundle of wires $\mathbb{Z}_+ \times \mathcal{V}$ are described by

$$\begin{cases} \text{Hilbert space} & \mathcal{H}_0 := \ell^2(\mathbb{Z}_+ \times \mathcal{V}) \equiv \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathcal{V}), \\ \text{Hamiltonian} & H_0 := h_0 \otimes \mathbf{1}_{\mathcal{V}}, \end{cases}$$

  where $h_0 := (T + T^*)/2$ with T being the shift operator.
- $\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [-1, 1]$.
- We fix a spectral representation for H_0

$$\mathscr{F}_0 H_0 \mathscr{F}_0^* = \int_{[-1,1]} \lambda \mathrm{d}\lambda, \quad \mathscr{F}_0 : \mathcal{H}_0 \to \int_{[-1,1]}^{\oplus} \mathfrak{h}_\lambda \mathrm{d}\lambda,$$

Fourier sine transform

with $\mathfrak{h}_\lambda \equiv \mathfrak{h} = \ell^2(\mathcal{V})$

Remark Discrete symmetries can be included by considering $H_0 \otimes \sigma$ with $\sigma = \sigma^* = \sigma^{-1}$, $\sigma \in M_L(\mathbb{C})$.

Generalized Childs' graph model

The perturbed system (wires + "graph")

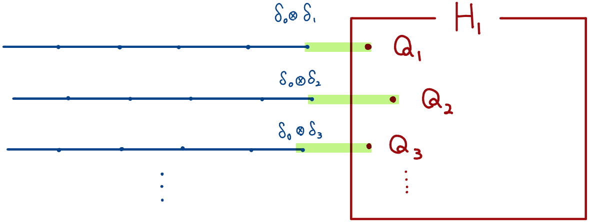
Let H_1 be a self-adjoint operator on a separable Hilbert space $\mathcal{H}_1$.

- The perturbed system is described by

$$\begin{cases} \text{Hilbert space} & \mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1, \\ \text{Hamiltonian} & H := \begin{pmatrix} H_0 & Q \\ Q^* & H_1 \end{pmatrix}. \end{cases}$$

- The interaction term $Q : \mathcal{H}_1 \to \mathcal{H}_0$ is a <u>bounded</u> operator of the form

$$Q = \sum_{n \in \mathcal{V}} |\delta_0 \otimes \delta_n\rangle\langle Q_n|, \qquad Q_n \in \mathcal{H}_1, Q_n \neq 0.$$



Mathematical Scattering Theory for Generalized Childs' Graph Model

Wave and scattering operators

Set $J : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ to be the inclusion operator. Then $HJ - JH_0 = Q^*$ and the existence of the WO can be justified with Cook's criterion.

Lemma

For any H_1, the WO $\boldsymbol{W}_\pm \equiv W_\pm(H, H_0; J)$ exist and are isometries.

- $\boldsymbol{W}_\pm H_0 = H \boldsymbol{W}_\pm$ (intertwining property) ,
- $\mathrm{Ran}(\boldsymbol{W}_\pm) \subset \mathcal{H}_{\mathrm{ac}}(H)$.

Remark

- By Kato-Rosenblum theorem, $\boldsymbol{W}_\pm$ are complete for **the Childs' graph model**, i.e. $\#\mathcal{V}, \dim \mathcal{H}_1 < \infty$.

The scattering operator S and the scattering matrix $S(\cdot)$ are defined by

$$S := \boldsymbol{W}_+^* \boldsymbol{W}_-, \qquad \mathscr{F}_0 S \mathscr{F}_0^* = \int_{[-1,1]}^{\oplus} S(\lambda) \mathrm{d}\lambda.$$

H_0 $\qquad$ $H_1 = H_0$

Stationary expressions

UNITARY?

We use the stationary approach to compute an explicit formula for S.

Resolvents

For $z \in \mathbb{C}$ with $\Im(z) \neq 0$, we set

$$R_0(z) := (H_0 - z)^{-1}, \quad R_1(z) = (H_1 - z)^{-1}, \quad R(z) = (H - z)^{-1}.$$

Lemma

The following stationary formulas hold:

$$\langle f, \boldsymbol{W}_\pm f_0 \rangle_{\mathcal{H}} = \int_{-1}^{1} \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \big\langle R(\lambda \pm i\varepsilon) f, J R_0(\lambda \pm i\varepsilon) f_0 \big\rangle_{\mathcal{H}} \mathrm{d}\lambda$$

for any $f \in \mathcal{H}_{\mathrm{ac}}(H)$ and f_0 in a dense subset of $\mathcal{H}_0$.

Reduction to one Hilbert space components

Let us consider the components

$$W_\pm := J^* \boldsymbol{W}_\pm,$$

which belong to $\mathcal{B}(\mathcal{H}_0)$.

1. One can compute S from the component W_- as

$$S = \operatorname*{s-lim}_{t\to\infty} \mathrm{e}^{itH_0} W_- \mathrm{e}^{-itH_0}.$$

2. It follows from the stationary formula that

$$\langle f_0', W_- f_0\rangle_{\mathcal{H}_0} = \int_{-1}^{1} \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi} \big\langle J^* R(\lambda - i\varepsilon) J f_0', R_0(\lambda - i\varepsilon) f_0 \big\rangle_{\mathcal{H}_0} \mathrm{d}\lambda$$

3. By Schur complement formula,

$$J^* R(z) J = (H_0 - Q R_1(z) Q^* - z)^{-1}.$$

4. The term $-QR_1(z)Q^*$ can be considered as an energy dependent potential localized on the boundary $\{0\} \times \mathcal{V} \subset \mathbb{Z}_+ \times \mathcal{V}$.

Explicit formula for the one Hilbert space component W_-

Theorem

Suppose that $\sigma(H_0) \cap \sigma(H_1)$ has Lebsegue measure 0. Then, one has

$$W_- = \mathbf{1} + \frac{1}{2}\big\{\tanh(A) - i\tanh(B)\cosh(A)^{-1}\big\}(S - \mathbf{1}) + K,$$

where

- $B := \tanh^{-1}(H_0)$,
- A *is a self-adjoint operator satisfying* $[B, A] = i\mathbf{1}$,
- *the remainder term* K *is generated by* S *and*

$$a(A)b(B) \qquad \textit{with } a, b \in \mathcal{S}(\mathbb{R})$$

Reamrk A corresponds to the generator of dilation groups in the continuous setting.

Remark Surprisingly, a completely same formula for the wave operator holds for Schrödinger operators on $\mathbb{Z}_+$ [I-Tsuzu, 2019].

Explicit formula for the scattering matrix $S(\lambda)$

Corollary

Suppose that $\sigma(H_0) \cap \sigma(H_1)$ has Lebsegue measure 0. Then, one has

$$S(\lambda) = \frac{1 + 2\mathrm{e}^{+i\theta} M(\lambda)}{1 + 2\mathrm{e}^{-i\theta} M(\lambda)}, \qquad \textit{a.e. } \lambda \in [-1, 1],$$

where

- $\lambda = \cos(\theta)$ *with* $\theta \in [0, \pi]$,
- $M(z)$ *is "the resolvent of H_1 restricted to the boundary", that is,*

$$M(z) := \Pi Q R_1(z) Q^* \Pi^*$$

with $\Pi : \mathcal{H}_0 \to \ell^2(\mathcal{V})$ defined by

$$\sum_{x=0}^{\infty} \delta_x \otimes \psi_x \mapsto \psi_0, \qquad \psi_x \in \ell^2(\mathcal{V}).$$

As a consequence, the scattering operator S is unitary.

Remarks

Corollary

If $\sigma(H_1) \cap \sigma(H_0)$ is a finite set, then the operator-valued function

$$[-1, 1] \ni \lambda \mapsto S(\lambda) \in \mathcal{B}\big(\ell^2(\mathcal{V})\big)$$

is smooth in norm.

Remark Our assumption is not optimal.

Example (attach wires everywhere)

Let $\mathcal{H}_1 = \ell^2(\mathcal{V})$ and $Q = \Pi^*$. Then, $Q^*\Pi^* = \Pi Q = \mathbf{1}_{\mathcal{V}}$ and therefore

$$S(\lambda) = \frac{H_1 - \lambda + 2\mathrm{e}^{+i\theta}}{H_1 - \lambda + 2\mathrm{e}^{-i\theta}}.$$

$\longrightarrow$ S is unitary even if H_1 has ac spectrum on $[-1, 1]$.

Topological Levinson's Theorem

C^*-algebraic framework cf. [Kellendonk-Richard,2007]

Assumption

Throughout this section we suppose that

- $\sigma(H_1) \cap \sigma(H_0)$ is a finite set,
- $N := \#\mathcal{V} < \infty$.

As a consequence, $(-1,1) \ni \lambda \mapsto S(\lambda) \in M_N(\mathbb{C})$ is smooth.

Let $\overline{\mathbb{R}} := [-\infty, \infty]$ be the two-point compactification of $\mathbb{R}$.

- We define a C^*-algebra $\mathcal{E}$ on $\mathcal{H}_0$ by

$$\mathcal{E} := C^*\Big(a(A)b(B)c \;\Big|\; a, b \in C\big(\overline{\mathbb{R}}\big),\ c \in M_N(\mathbb{C})\Big),$$

- The set of compact operators on $\mathcal{H}_0$ is also expressed as

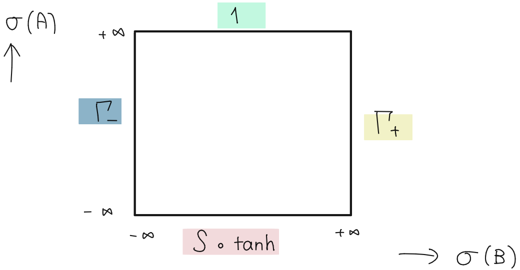
$$\mathcal{K}(\mathcal{H}_0) = C^*\Big(a(A)b(B)c \;\Big|\; a, b \in C_0(\mathbb{R}),\ c \in M_N(\mathbb{C})\Big).$$

- $W_- \in \mathcal{E}$ and $\underset{\sim}{K} \in \mathcal{K}$.

Quotient algebra

- $\mathcal{E}/\mathcal{K}(\mathcal{H}_0) \cong C(\square) \otimes M_N(\mathbb{C})$ with $\square = \partial\big(\overline{\mathbb{R}} \times \overline{\mathbb{R}}\big) \cong \mathbb{T}$.
- The quotient map $\pi : \mathcal{E} \to C\big(\square; M_N(\mathbb{C})\big)$ is given by

$$\pi\big(a(A)b(B)\big) = \big(a(-\infty)b, b(+\infty)a, a(+\infty)b, b(-\infty)a\big).$$



- $\pi(W_-) =: \Gamma \equiv (S \circ \tanh, \Gamma_+, 1, \Gamma_-)$,
- The functions $\Gamma_\pm$ are given by

$$\Gamma_\pm(t) = \mathbf{1} - E_\pm - \big(\tanh(t) \mp i\cosh(t)^{-1}\big)E_\pm, \qquad t \in \mathbb{R},$$

where $E_\pm$ are projections (onto the set of resonances at ± 1).

Relative topological Levinson's theorem

We get the following index theorem not for the wave operator $\boldsymbol{W}_-$ but for the one Hilbert space component W_-:

Theorem

Under Assumption (A), Γ *is a unitary element in* $C(\square) \otimes M_N(\mathbb{C})$*.*

Moreover, one has

$$\mathrm{Wind}\big(\det(\Gamma)\big) = -\,\mathrm{Index}(W_-)$$

Remark With the notation $s(\lambda) = \det(S(\lambda))$, the l.h.s. is given by

$$\mathrm{Wind}\big(\det(\Gamma)\big) = \frac{1}{2\pi i}\int_{-1}^{1} s'(\lambda)s(\lambda)^* \mathrm{d}\lambda - \delta$$

Topological interpretation of Childs-Levinson theorem

Consider Childs' graph model (i.e. $\#\mathcal{V}, \dim \mathcal{H}_1 < \infty$).

- $J : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ is Fredholm with $\mathrm{Index}(J) = -\dim \mathcal{H}_1$.
- Since $\boldsymbol{W}_- = JW_- +$ (finite-rank), $\boldsymbol{W}_-$ is also Fredholm.
- Combining this with the completeness

$$\mathcal{H}_{\mathrm{sc}}(H) = 0 \quad \text{and} \quad \mathrm{coker}(\boldsymbol{W}_-) = \mathcal{H}_{\mathrm{pp}}(H).$$

Then, the r.h.s. of topological Levinson's theorem is given by

$$\begin{aligned} -\mathrm{Index}(W_-) &= -\mathrm{Index}(J^*\boldsymbol{W}_-) \\ &= -\mathrm{Index}(\boldsymbol{W}_-) + \mathrm{Index}(J) \\ &= \dim \mathrm{coker}(\boldsymbol{W}_-) - \dim \mathcal{H}_1 \\ &= \dim \mathcal{H}_{\mathrm{pp}}(H) - \#\{\text{vertices of } G_1\} \end{aligned}$$

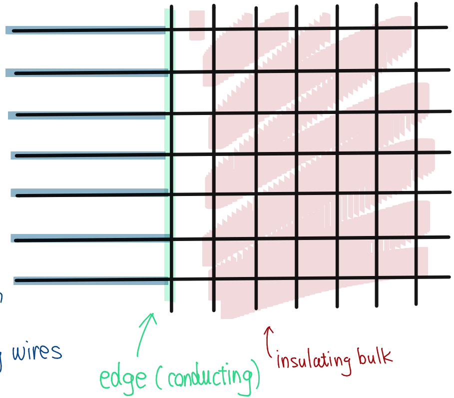
Conclusion Levinson's theorem for Childs' graph model is also a realization of the index theorem!

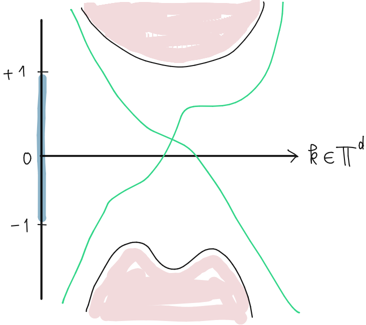
Work in progress: Bulk-edge correspondence via two spaces scattering

The model

The model (Wires + $(1+d)$-dim insulator)

- $H_{\text{bulk}} \curvearrowright \ell^2(\mathbb{Z}^{1+d})$, periodic with a gap on $\sigma(H_0)$,
- $H_1 := H_{\text{edge}} \curvearrowright \ell^2(\mathbb{Z}_+ \times \mathbb{Z}^d)$, $\mathbb{Z}^d$-periodic with gapless edge states,
- Wires are also attached periodically to the boundary $\mathcal{V} = \mathbb{Z}^d$.





Unitarity of S for edge Hamiltonians

By the Floquet-Bloch decomposition w. r. t. the $\mathbb{Z}^d$-action,

$$H_0 \cong \int_{\mathbb{T}^d} H_0(k)\mathrm{d}k, \quad H_1 \cong \int_{\mathbb{T}^d} H_1(k)\mathrm{d}k, \quad H \cong \int_{\mathbb{T}^d} H(k)\mathrm{d}k,$$

$$\boldsymbol{W}_\pm \cong \int_{\mathbb{T}^d} \boldsymbol{W}_\pm(k)\mathrm{d}k, \quad W_\pm \cong \int_{\mathbb{T}^d} W_\pm(k)\mathrm{d}k, \quad S \cong \int_{\mathbb{T}^d} S(k)\mathrm{d}k.$$

- $H_0(k) = h_0$.
- For each $k \in \mathbb{T}^d$, $\sigma(H_1(k)) \cap \sigma(H_0)$ is finite.
- The scattering operator $S(k)$ for the pair $(H_0(k), H(k))$ is unitary.

$\longrightarrow$ S **is unitary for edge states.**

(Edge states are absorbed into the conducting wires)

Remark For $H_1 = H_{\mathrm{edge}}$, H_0 can be replaced with $(1+d)$-dim half-space Laplacian $\Delta_{\mathbb{Z}_+ \times \mathbb{Z}^d}$ (metal-insulator junction). Indeed, one reduces the system to the generalised Childs' graph model with $\#\mathcal{V} = 1$ for each fixed quasi-momentum $k \in \mathbb{T}^d$.

Work in progress: Levinson and bulk-edge correspondence

Levinson's theorem in disordered system

Topological Levinson's theorem for periodic edge Hamiltonians can be formulated by using trace per unit volume along the boundary $\mathbb{Z}^d$.

$\longrightarrow$ Is the scattering operator S still unitary for disordered systems?

Bulk-edge correspondence via scattering theory

The Chern number of the reflection coefficient $R^{\lambda+i\varepsilon}$, which also related to the scattering part of H, is equal to the bulk and edge invariants [Schulz-Baldes & Toniolo '21].

$\longrightarrow$ (CONJECTURE) For $\lambda \in \sigma(H_0)$, at least in K-theory level

$$S(\lambda) = R^{\lambda+i0} \qquad ?$$

Conclusion

- For fairly general class of Hamiltonians H_1 we obtained

$$(H_1, Q) \mapsto \boldsymbol{W}_{\pm}, W_{\pm}, S.$$

- Topological Levinson's theorem by considering the one Hilbert space component $W_- = J^* \boldsymbol{W}_-$.
- Potential applications to the bulk-edge correspondence.

References

A. M. Childs, D. J. Strouse, *Levinson's theorem for graphs*, J. Math. Phys. 52 (2011).

A. M. Childs, D. Gosset, *Levinson's theorem for graphs II*, J. Math. Phys. 52 (2012).

H. Inoue, N. Tsuzu, *Schrödinger Wave Operators on the Discrete Half-Line*, Integr. Equ. Oper. Theory **91**, 1-12 (2019).

J. Kellendonk, S. Richard, *The topological meaning of Levinson's theorem, half-bound states included*, J. Phys. A: Math. Theo. **41**, (2008).

H. Schulz-Baldes, D. Toniolo, *Dimensional reduction and scattering formulation for even topological invariants*, Commun. Math. Phys. **381**, (2021).