Strong radiation condition and stationary scattering theory for Stark operators

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1. Known results for perturbed Laplacian

Let $V(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ for simplicity, and $P = -\Delta + V$. Set

$$\mathcal{B}^* = \Big\{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \,\Big|\, \sup_{\rho>0} \rho^{-1} \int_{|x|\le\rho} |\psi(x)|^2 \,\mathrm{d}x < \infty\Big\}.$$

For $\lambda > 0$ and $\psi \in \mathcal{B}^*$, we define $\mathcal{F}^\pm(\lambda, r)\psi \in L^2(\mathbb{S}^{d-1})$ as

$$\big(\mathcal{F}^\pm(\lambda, r)\psi\big)(\omega) = \pi^{-1/2}\lambda^{1/4} r^{(d-1)/2} \mathrm{e}^{\mp \mathrm{i}K(r\omega,\lambda)} \big(R(\lambda \pm \mathrm{i}0)\psi\big)(r\omega),$$

where $r = |x|, \omega \in \mathbb{S}^{d-1}$ and $K(r\omega, \lambda) = \sqrt{\lambda}|x| - \frac{\pi}{4}(d-3)$.
K is a 'good' solution to the eikonal equation: $|\nabla K|^2 + V - \lambda = 0$.
Then there exist the limits

$$\mathcal{F}^\pm(\lambda)\psi := \lim_{r\to\infty} \mathcal{F}^\pm(\lambda, r)\psi,$$

and one has $\|\mathcal{F}^\pm(\lambda)\psi\|^2 = \frac{1}{2\pi\mathrm{i}}\langle\psi, \big(R(\lambda + \mathrm{i}0) - R(\lambda - \mathrm{i}0)\big)\psi\rangle$.

$\mathcal{F}^\pm(\lambda)$ are bounded and are called *stationary wave operators*.

Known results for perturbed Laplacian

The adjoints $\mathcal{F}^{\pm}(\lambda)^*$ are eigenoperators: For any $\xi \in L^2(\mathbb{S}^{d-1})$

$$(P-\lambda)\mathcal{F}^{\pm}(\lambda)^*\xi = 0.$$

We also mention an application of stationary scattering theory. Under same notations as above, it holds that

(1) For any function $\xi_- \in L^2(\mathbb{S}^{d-1})$, there exist a function $\xi_+ \in L^2(\mathbb{S}^{d-1})$ and a solution ϕ of $(P-\lambda)\phi = 0$ with asymptotic

$$\phi(x) - \xi_+(\omega)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|} + \xi_-(\omega)\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}|x|} \in \mathcal{B}_0^*; \qquad (1.1)$$

$$\mathcal{B}_0^* = \Big\{\psi \in \mathcal{B}^* \,\Big|\, \lim_{\rho\to\infty} \rho^{-1} \int_{|x|\le\rho} |\psi(x)|^2 \,\mathrm{d}x = 0\Big\}.$$

Moreover one has $\|\xi_+\| = \|\xi_-\|$.

(2) Every $\phi \in \mathcal{B}^*$ satisfying $(P-\lambda)\phi = 0$ has asymptotic (1.1) with some functions $\xi_\pm \in L^2(\mathbb{S}^{d-1})$.

References

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[2] Y. Gâtel, D. Yafaev, *On solutions of the Schrödinger equation with radiation conditions at infinity: the long-range case*, Ann. Inst. Fourier, Grenoble. 49, 5, 1581–1602 (1999).

They consider $P = -\Delta + V$, where $V \in C^3(\mathbb{R}^d)$ and for some $\delta > 0$

$$\partial^\alpha V(x) = \mathcal{O}(|x|^{-|\alpha|-\delta}) \;\text{ as }\; |x| \to \infty \;\; (\alpha \in \mathbb{N}_0^d,\; |\alpha| \leq 3).$$

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2. Setting
Stark operator

We introduce a perturbed Stark operator on the Euclidean space $\mathbb{R}^d, d \geq 2$, which is a quantum mechanical model of a charged d-dimensional particle subject to a constant electric field and one-body interaction $q = q(\cdot)$ decaying at infinity.

We split and write the coordinate of $\mathbb{R}^d$ as

$$(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}, \quad x = x_1,\ y = (y_2, \ldots, y_d),$$

and assume that the constant electric field points in the x-direction. Then perturbed Stark operator is given by

$$H = H_0 + q; \quad H_0 = \tfrac{1}{2}p^2 - x, \quad p = -\mathrm{i}\nabla.$$

By Condition A introduced later, we may regard H as a self-adjoint operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$.

Parabolic coordinates

We use a slightly modified parabolic coordinate $(f, g) = (f, g_2, \dots, g_d)$ satisfying

$$x = \tfrac{1}{2}(f^2 - g^2), \quad y = fg$$

on a certain region.

Let $\breve{f} \in C^\infty(\mathbb{R})$ be a convex function such that $\breve{f}(t) = 1$ for $t \le 1/2$ and $\breve{f}(t) = t$ for $t \ge 2$. We define $f \in C^\infty(\mathbb{R}^d)$ as

$$f(x, y) = \breve{f}(r + x)^{1/2}; \quad r = (x^2 + y^2)^{1/2}.$$

The other 'parabolic variables' are defined as

$$g = f^{-1}y, \quad \text{or} \quad g_i = f^{-1}y_i \ \text{ for } i = 2, \dots, d.$$

Condition on q

Condition A

The perturbation q splits into two real-valued measurable functions as $q = q_1 + q_2$ such that:

1. $q_1 \in C^1(\mathbb{R}^d)$, and there exist $\delta \in (0, 1]$ and $C > 0$ such that for any $|\alpha| \le 1$

$$|\partial^{\alpha}_{(x,y)} q_1| \le C f^{-1-\delta-2|\alpha|}.$$

2. $\operatorname{supp} q_2$ is compact, and $q_2(-\Delta + 1)^{-1}$ is a compact operator on $\mathcal{H}$.

In addition, a function $\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ has to be identically zero on $\mathbb{R}^d$ if it satisfies

i) $(H - \lambda)\phi = 0$ for some $\lambda \in \mathbb{R}$ in the distributional sense,

ii) $\phi = 0$ on a non-empty open subset of $\mathbb{R}^d$.

Besov spaces

Let $\mathcal{B} = \mathcal{B}(f)$, $\mathcal{B}^* = \mathcal{B}^*(f)$ and $\mathcal{B}^*_0 = \mathcal{B}^*_0(f)$ be the *Besov spaces* with respect to the multiplication operator by the function f, i.e.

$$\mathcal{B} = \Big\{ \psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \,\Big|\, \sum_{n\in\mathbb{N}_0} 2^{n/2} \|F_n\psi\|_{L^2} < \infty \Big\},$$
$$\mathcal{B}^* = \Big\{ \psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \,\Big|\, \sup_{n\in\mathbb{N}_0} 2^{-n/2} \|F_n\psi\|_{L^2} < \infty \Big\},$$
$$\mathcal{B}^*_0 = \Big\{ \psi \in \mathcal{B}^* \,\Big|\, \lim_{n\to\infty} 2^{-n/2} \|F_n\psi\|_{L^2} = 0 \Big\},$$

where $F_n = 1_{\{2^n \le f < 2^{n+1}\}}$ is the characteristic function of the set specified by $2^n \le f < 2^{n+1}$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We remark that the inclusions

$$L^2_s \subsetneq \mathcal{B} \subsetneq L^2_{1/2} \subsetneq L^2 \subsetneq L^2_{-1/2} \subsetneq \mathcal{B}^*_0 \subsetneq \mathcal{B}^* \subsetneq L^2_{-s},$$

where $L^2_s = f^{-s} L^2(\mathbb{R}^d)$, hold for any $s > 1/2$.

WKB-approximation

Let $\Sigma = L^2(\mathbb{R}^{d-1}) = L^2(\mathbb{R}^{d-1}_g; \mathrm{d}g)$. For any $\kappa > 0$ we let $\chi(\cdot < \kappa)$ be a smooth cut-off function χ on $\mathbb{R}$ such that

$$0 \leq \chi \leq 1, \quad \operatorname{supp} \chi \subseteq (-\infty, \kappa), \quad \chi(t) = 1 \text{ for } t \leq 3\kappa/4.$$

In addition, we introduce $\chi^\perp(\cdot < \kappa) = 1 - \chi(\cdot < \kappa)$.
Then for any $\lambda \in \mathbb{R}$ and $\xi \in \Sigma$ we set

$$\phi^\pm_\lambda[\xi](f, g) = \omega_\pm^{-1} \chi^\perp(f < 2) J(f, g)^{1/2} \, \mathrm{e}^{\pm \mathrm{i}\theta_\lambda(f)} \, \xi(\pm g) \in \mathcal{B}^*,$$

where $\omega_\pm = (2\pi)^{1/2} \mathrm{e}^{\pm \mathrm{i}\pi d/4}$ and

$$J(f, g) = f^{2-d}(f^2 + g^2)^{-1}, \quad \theta_\lambda(f) = \tfrac{1}{3} f^3 + \lambda f.$$

J is the Jacobian associated with the coordinate change between (x, y) and (f, g) for $r + x > 2$, and θ_λ is an approximate solution to the eikonal equation in the sense that for g bounded

$$\tfrac{1}{2} |\nabla \theta_\lambda|^2 - x - \lambda = \mathcal{O}(f^{-2}).$$

WKB-approximation

The functions $\phi^{\pm}_{\lambda}[\xi]$ constitute WKB-approximations of solutions to the eigenequation with purely outgoing/incoming radiation conditions (2.1) below, respectively.

Let us introduce

$$\partial_f = 2r(\nabla f)\cdot\nabla, \quad p_f = -\mathrm{i}\partial_f,$$
$$B = \operatorname{Re} p_f = 2r(\nabla f)\cdot p - \mathrm{i}\operatorname{div}(r\nabla f).$$

∂_f coincide with the coordinate derivative $\frac{\partial}{\partial f}$ for $r + x > 2$, although does not coincide with that for $r + x \le 2$.
For $r + x > 2$ we can see $BJ^{1/2} = 0$, and hence the *outgoing/incoming radiation conditions*

$$\big(B \mp (\partial_f \theta_\lambda)\big)\phi^{\pm}_{\lambda}[\xi] = 0, \tag{2.1}$$

respectively. Moreover $\phi^{\pm}_{\lambda}[\xi]$ for $\xi \in C^{\infty}_{\mathrm{c}}(\mathbb{R}^{d-1})$ are *approximate* generalized eigenfunctions in the sense that

$$\psi^{\pm}_{\lambda}[\xi] := (H-\lambda)\phi^{\pm}_{\lambda}[\xi] \in \mathcal{B}.$$

2. Main results

Radiation condition

In the following we always assume Condition A.

For any $\lambda \in \mathbb{R}$ we introduce 'gamma observable' as

$$\gamma(\lambda) = p \mp \nabla\theta_\lambda, \quad \tilde{\gamma} = f^{-1}g$$

and the derived observable

$$\gamma_\parallel(\lambda) = \mathrm{Re}\bigl(2r(\nabla f)\cdot\gamma(\lambda)\bigr) = B \mp \partial_f\theta_\lambda,$$

We also refer these quantities as *radiation operators*.

Proposition 3.1

Let $I \subseteq \mathbb{R}$ be a compact interval, and let $s \in (0, 1+\delta)$. Then there exists $C > 0$ such that for any $i, j = 1, \ldots, d$ and any $\phi = R(\lambda \pm \mathrm{i}0)\psi$ with $\lambda \in I$ and $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$

$$\begin{aligned}&\|f^s\gamma_i(\lambda)\gamma_j(\lambda)\phi\|_{L^2_{-1/2}} + \|f^{(s+2)/2}\tilde{\gamma}^2\phi\|_{\mathcal{B}^*} + \|f^s\gamma_\parallel(\lambda)\phi\|_{L^2_{-1/2}}\\ &\le C\|f^{s+2}\psi\|_{L^2_{1/2}}.\end{aligned}$$

Stationary wave operators

For any $\psi \in \mathcal{B}$ we introduce

$$\mathcal{R}^{\pm}_{\lambda}\psi(f,g) = \omega_{\pm} J(f,g)^{-1/2} \mathrm{e}^{\mp \mathrm{i}\theta_\lambda(f)} \big(R(\lambda \pm \mathrm{i}0)\psi\big)(f,g).$$

Then we expect that there exist the limits

$$\Sigma\text{-}\lim_{f\to\infty} \mathcal{R}^{\pm}_{\lambda}\psi(f,\cdot) := \lim_{f\to\infty} \mathcal{R}^{\pm}_{\lambda}\psi(f,\cdot) \text{ in } \Sigma. \tag{3.1}$$

By Proposition 3.1 we can verify the existence of the limits (3.1) for $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$. In the later argument and the case of $\psi \in \mathcal{B}$, we take the limits in the averaged sense. For any vector-valued function η of f, let us use the notation

$$⨍_{\rho} \eta(f)\,\mathrm{d}f = \rho^{-1}\int_{\rho}^{2\rho} \eta(f)\,\mathrm{d}f, \quad \rho > 0.$$

Stationary wave operators

Theorem 3.2

(1) *For any* $\lambda \in \mathbb{R}$ *and* $\psi \in \mathcal{B}$ *the following averaged limits exist,*

$$\mathcal{F}^{\pm}(\lambda)\psi := \pm(2\pi\mathrm{i})^{-1} \ \underset{\rho\to\infty}{\Sigma\text{-}\lim} ⨍_{\rho} \mathcal{R}_{\lambda}^{\pm}\psi(f, \pm\cdot)\,\mathrm{d}f.$$

(2) *The mappings* $\mathbb{R} \times \mathcal{B} \ni (\lambda, \psi) \mapsto \mathcal{F}^{\pm}(\lambda)\psi \in \Sigma$ *are continuous.*

(3) *For any* $\lambda \in \mathbb{R}$ *the operators* $\mathcal{F}^{\pm}(\lambda) \colon \mathcal{B} \to \Sigma$ *are surjective.*

(4) *For any* $\lambda \in \mathbb{R}$ *and* $\psi \in \mathcal{B}$

$$\|\mathcal{F}^{\pm}(\lambda)\psi\|_{\Sigma}^{2} = \langle \psi, \delta(H-\lambda)\psi \rangle,$$

where $\delta(H-\lambda) = \pi^{-1}\,\mathrm{Im}\,R(\lambda+\mathrm{i}0)$.

The operators $\mathcal{F}^{\pm}(\lambda) \in \mathcal{L}(\mathcal{B}, \Sigma)$ are called the *stationary wave operators*, and their surjectivity is the *stationary completeness*.

Generalized Fourier transforms

Using the stationary wave operators $\mathcal{F}^{\pm}(\lambda)$ we introduce

$$\mathcal{F}^{\pm} = \int_{\mathbb{R}}^{\oplus} \mathcal{F}^{\pm}(\lambda)\,\mathrm{d}\lambda, \quad \widetilde{\mathcal{H}} = L^2(\mathbb{R}, \mathrm{d}\lambda; \Sigma),$$

and let M_λ denote the multiplication operator by λ on $\widetilde{\mathcal{H}}$.
We note that $\mathcal{F}^{\pm} : \mathcal{B} \ni \psi \mapsto \big(\mathcal{F}^{\pm}\psi\big)(\cdot) = \mathcal{F}^{\pm}(\cdot)\psi \in C(\mathbb{R}; \Sigma)$.

Theorem 3.3

The operators $\mathcal{F}^{\pm}$ extend as unitary operators $\mathcal{F}^{\pm} \colon \mathcal{H} \to \widetilde{\mathcal{H}}$. Moreover $\mathcal{F}^{\pm}$ diagonalize H, that is,

$$\mathcal{F}^{\pm} H = M_\lambda \mathcal{F}^{\pm}.$$

In particular, H and M_λ are unitarily equivalent.

Stationary wave matrices and Scattering matrix

The adjoints $\mathcal{F}^{\pm}(\lambda)^* \in \mathcal{L}(\Sigma, \mathcal{B}^*)$ are called the *stationary wave matrices*. We let

$$\mathcal{E}_\lambda = \{\phi \in \mathcal{B}^* \,|\, (H - \lambda)\phi = 0\},$$

and call the elements of $\mathcal{E}_\lambda$ *minimal* generalized eigenfunctions in the sense that there are no generalized eigenfunctions in the slightly smaller space $\mathcal{B}_0^*$. Then the wave matrices $\mathcal{F}^{\pm}(\lambda)^* \colon \Sigma \to \mathcal{E}_\lambda \, (\subseteq \mathcal{B}^*)$ are topological linear isomorphisms. Thus $\mathcal{F}^{\pm}(\lambda)^*$ are eigenoperators of H with eigenvalue $\lambda \in \mathbb{R}$.

By Theorem 3.2, we can define the *stationary scattering matrix* $S(\lambda) \in \mathcal{L}(\Sigma)$ for $\lambda \in \mathbb{R}$ as the unitary operator obeying

$$\mathcal{F}^{+}(\lambda)\psi = S(\lambda)\mathcal{F}^{-}(\lambda)\psi \quad \text{for any } \psi \in \mathcal{B}.$$

We note that the mapping $\mathbb{R} \ni \lambda \mapsto S(\lambda) \in \mathcal{L}(\Sigma)$ is strongly continuous. This follows from (2) of Theorem 3.2.

Asymptotics of generalized eigenfunctions

Theorem 3.4

(1) *For any one of* $\xi_\pm \in \Sigma$ *or* $\phi \in \mathcal{E}_\lambda$ *the two other quantities in* $\{\xi_-, \xi_+, \phi\}$ *uniquely exist such that*

$$\phi - \phi_\lambda^+[\xi_+] - \phi_\lambda^-[\xi_-] \in \mathcal{B}_0^*. \tag{3.2}$$

(2) *The correspondences in* (3.2) *are given by the formulas*

$$\phi = \mathcal{F}^\pm(\lambda)^*\xi_\pm, \qquad \xi_+ = S(\lambda)\xi_-,$$

$$\xi_\pm = \omega_\pm \ \Sigma\text{-}\lim_{\rho\to\infty} \fint_\rho \tfrac{1}{2}\big(J^{-1/2}\,\mathrm{e}^{\mp\mathrm{i}\theta_\lambda}\big(1 \pm (f\sqrt{2r})^{-1}p_f\big)\phi\big)(f, \pm\cdot)\,\mathrm{d}f.$$

(3) *For any* $\xi_\pm \in \Sigma$ *and* $\phi \in \mathcal{E}_\lambda$ *satisfying* (3.2)

$$\|\xi_\pm\|_\Sigma = \pi^{1/2} \lim_{n\to\infty} 2^{-n/2}\|F_n\phi\|_{L^2}.$$

4. Classical mechanics version of Proposition 3.1

Consider a classical Hamiltonian for $(x, y; \eta, \zeta) \in T^*\mathbb{R}^d \cong \mathbb{R}^{2d}$

$$H^{\mathrm{cl}}(x, y, \eta, \zeta) = \tfrac{1}{2}(\eta^2 + \zeta^2) - x + q_1(x, y). \tag{4.1}$$

The associated Hamilton equations are

$$\dot{x} = \eta, \quad \dot{y} = \zeta, \quad \dot{\eta} = 1 - \partial_x q_1, \quad \dot{\zeta} = -\partial_y q_1. \tag{4.2}$$

First we let $q_1 \equiv 0$, and then we can explicitly solve (4.2).
A solution to (4.2) with initial data $(x_0, y_0; \eta_0, \zeta_0) \in T^*\mathbb{R}^d$ is

$$x = \tfrac{1}{2}t^2 + t\eta_0 + x_0, \quad y = t\zeta_0 + y_0, \quad \eta = t + \eta_0, \quad \zeta = \zeta_0.$$

In particular we have

$$f - t = \mathcal{O}(1) \ \text{ as } t \to \infty.$$

This implies that we may regard the quantity f as an 'effective time', which allows us to effectively rewrite time-dependent observables by time-independent ones.

Classical gamma observables

Moreover we have, as $t \to \infty$,

$$\eta - f = \mathcal{O}(f^{-1}), \quad \zeta - g = \mathcal{O}(f^{-1}), \quad f^{-1}g = \mathcal{O}(f^{-1}). \tag{4.3}$$

This means that (f, g) is comparable to the momentum (η, ζ).
Let us consider the case of $\lambda = 0$, and set

$$\gamma^{\mathrm{ex}} = (\eta, \zeta) - \nabla\theta^{\mathrm{ex}}, \quad \tilde{\gamma}^{\mathrm{ex}} = f^{-1}g, \quad \gamma_{\|}^{\mathrm{ex}} = (\nabla\theta^{\mathrm{ex}}) \cdot \gamma^{\mathrm{ex}},$$

where

$$\theta^{\mathrm{ex}}(x, y) = \tfrac{4}{3}\sqrt{x + (x^2 - y^2)^{1/2}}\big(x - \tfrac{1}{2}(x^2 - y^2)^{1/2}\big); \quad x > |y|$$

is an exact solution to the eikonal equation $\frac{1}{2}|\nabla\theta|^2 - x = 0$.
Then by (4.3) we easily have

$$\gamma^{\mathrm{ex}} = \mathcal{O}(f^{-1}), \quad \tilde{\gamma}^{\mathrm{ex}} = \mathcal{O}(f^{-1}),$$

and further

$$\gamma_{\|}^{\mathrm{ex}} = H^{\mathrm{cl}} - \tfrac{1}{2}(\gamma^{\mathrm{ex}})^2 = -\tfrac{1}{2}(\gamma^{\mathrm{ex}})^2 = \mathcal{O}(f^{-2}).$$

Perturbed case

For simplicity, let us consider the case of zero-energy, and set

$$\gamma = \gamma^{\mathrm{cl}} = (\eta, \zeta) - \nabla\theta_0, \quad \tilde{\gamma} = \tilde{\gamma}^{\mathrm{cl}} = f^{-1}g,$$
$$\gamma_{\|} = \gamma_{\|}^{\mathrm{cl}} = 2r(\nabla f)\cdot\gamma^{\mathrm{cl}}.$$

Lemma 4.1

Let (x, y, η, ζ) be a zero-energy forward scattering orbit for the perturbed Stark Hamiltonian (4.1)*. Then for any $s \in (0, 1+\delta)$*

$$\gamma^2 = \mathcal{O}\big(f^{-s}\big), \quad \tilde{\gamma}^2 = \mathcal{O}\big(f^{-(s+2)/2}\big), \quad \gamma_{\|} = \mathcal{O}\big(f^{-s}\big), \qquad (4.4)$$

as $t \to \infty$.

We remark that by the identity (holding for $r + x > 2$)

$$0 = H^{\mathrm{cl}} = \tfrac{1}{2}\gamma^2 + \tfrac{1}{2}f^2 r^{-1}\gamma_{\|} + \tfrac{1}{4}f^4 r^{-1}\tilde{\gamma}^4 + q_1, \qquad (4.5)$$

the bound (4.4) is sharp under Condition A.

Proof of Lemma 4.1

We prove Lemma 4.1 without using the time parameter t as much as possible, and then we can generalize the scheme to quantum mechanics.

We note that x is bounded from below by conservation of energy low, and since (x, y, η, ζ) is a scattering orbit, we may assume

$$f \geq c_1|g|, \quad r + x > 2$$

for some $c_1 > 0$ and for large $t \geq 0$. In particular, there exists $C_1 > 0$ such that for large $t \geq 0$

$$\tfrac{1}{2}f^2 \leq r \leq C_1 f^2.$$

By the identity

$$0 = H^{\mathrm{cl}} = \tfrac{1}{2}\gamma^2 + \tfrac{1}{2}f^2 r^{-1}\gamma_{\|} + \tfrac{1}{4}f^4 r^{-1}\tilde{\gamma}^4 + q_1, \tag{4.5}$$

it suffices to show that

$$\gamma_{\|} = \mathcal{O}(f^{-s}).$$

Proof of Lemma 4.1

To prove $\gamma_{\|} = \mathcal{O}(f^{-s})$, we compute the time-derivative of $(f^s \gamma_{\|})^2$:

$$D\big((f^s\gamma_{\|})^2\big) = 2sf^{2s-1}\gamma_{\|}^2(Df) + 2f^{2s}\gamma_{\|}\big(D\gamma_{\|}\big), \tag{4.6}$$

where $D = \frac{\mathrm{d}}{\mathrm{d}t}$, and show that this quantity is almost negative.
We bound the first term to the right of (4.6) as, by using (4.5),

$$\begin{aligned}
2sf^{2s-1}\gamma_{\|}^2(Df) &= 2sf^{2s-1}\gamma_{\|}^2(\nabla f)\cdot(\eta,\zeta) \\
&= sf^{2s-1}r^{-1}\gamma_{\|}^3 + sf^{2s+1}r^{-1}\gamma_{\|}^2 \\
&\le sf^{2s+1}r^{-1}\gamma_{\|}^2 + C_2 f^{2s-4-\delta}\gamma_{\|}^2 \\
&\le sf^{2s+1}r^{-1}\gamma_{\|}^2 + C_3 f^{2s-2-\delta}\gamma^2 \\
&\le sf^{s+1}r^{-1}\gamma_{\|}^2 - \tfrac{1}{2}C_3 f^{2s-\delta}r^{-1}\gamma_{\|} + C_4 f^{2s-3-2\delta} \\
&\le (s+\epsilon)f^{2s+1}r^{-1}\gamma_{\|}^2 + C_5 f^{2s-3-2\delta}.
\end{aligned}$$

Here $\epsilon \in (0, 1-s/2)$ is a fixed constant.

Proof of Lemma 4.1

Since $D\gamma_{\|}$ can be computed as

$$D\gamma_{\|} = 2\gamma\cdot(\nabla r\nabla f)\gamma + \tfrac{1}{4}fg^4r^{-2} + \tfrac{1}{2}f^3r^{-2}\gamma_{\|} \\ - fr^{-1}\gamma_{\|} - 2r(\nabla f)\cdot(\nabla q_1),$$

we have the bounds

$$-fr^{-1}\gamma_{\|} - C_6f^{-2-\delta} \le D\gamma_{\|} \le C_6f^{-1}\big(\gamma^2 + f^2\tilde{\gamma}^4 + |\gamma_{\|}| + f^{-1-\delta}\big).$$

Then the second term of (4.6) is bounded as, by using (4.5) again,

$$\begin{aligned}
&2f^{2s}\gamma_{\|}\big(D\gamma_{\|}\big)\\
&= -f^{2s-2}r\big(2\gamma^2 + f^4r^{-1}\tilde{\gamma}^4\big)\big(D\gamma_{\|}\big) - 2q_1f^{2s-2}r\big(D\gamma_{\|}\big)\\
&\le f^{2s-1}\big(2\gamma^2 + f^4r^{-1}\tilde{\gamma}^4\big)\gamma_{\|} + C_7f^{2s-2-\delta}\big(\gamma^2 + f^2\tilde{\gamma}^4\big)\\
&\quad + C_7f^{2s-2-\delta}\big(\gamma^2 + f^2\tilde{\gamma}^4 + |\gamma_{\|}| + f^{-1-\delta}\big)\\
&\le -2f^{2s+1}r^{-1}\gamma_{\|}^2 + C_8f^{2s-2-\delta}\big(\gamma^2 + f^2\tilde{\gamma}^4 + |\gamma_{\|}| + f^{-1-\delta}\big)\\
&\le -2f^{2s+1}r^{-1}\gamma_{\|}^2 + C_9f^{2s-2-\delta}|\gamma_{\|}| + C_9f^{2s-3-2\delta}\\
&\le -(2-\epsilon)f^{2s+1}r^{-1}\gamma_{\|}^2 + C_{10}f^{2s-3-2\delta}.
\end{aligned}$$

Proof of Lemma 4.1

By summing up the bounds

$$\begin{aligned} 2sf^{2s-1}\gamma_{\|}^2(Df) &\leq (s+\epsilon)f^{2s+1}r^{-1}\gamma_{\|}^2 + C_5 f^{2s-3-2\delta}, \\ 2f^{2s}\gamma_{\|}\big(D\gamma_{\|}\big) &\leq -(2-\epsilon)f^{2s+1}r^{-1}\gamma_{\|}^2 + C_{10}f^{2s-3-2\delta}, \end{aligned}$$

we have (noting $\epsilon \in (0, 1-s/2)$)

$$\begin{aligned} D\big((f^s\gamma_{\|})^2\big) &\leq -(2-s-2\epsilon)fr^{-1}(f^s\gamma_{\|})^2 + C_{11}f^{2s-3-2\delta} \\ &\leq -c_2 f^{-1}\big[(f^s\gamma_{\|})^2 - C_{12}f^{2s-2-2\delta}\big]. \end{aligned} \tag{4.7}$$

Now let us show that $(f^s\gamma_{\|})^2$ is bounded as $t \to \infty$. If not, we can find a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad (f^s\gamma_{\|})^2(t_n) \geq n, \quad D\big((f^s\gamma_{\|})^2\big)(t_n) \geq 0.$$

However this contradicts (4.7), since $f(t_n) \to \infty$ as $n \to \infty$ and $s \in (0, 1+\delta)$. Thus $(f^s\gamma_{\|})^2$ is bounded, and hence $\gamma_{\|} = \mathcal{O}(f^{-s})$.

References

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Thank you for your attention.