Asymptotic behavior of the Spectral Shift Function for a discrete Dirac-type operator in $\mathbb{Z}^2$

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Graph structure

Define the set of vertices $\mathcal{V} = \mathbb{Z}^2$.
The set of oriented edges is $\mathcal{A} = \{(x,y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : y = x \pm \delta_i\}$, where $\{\delta_1, \delta_2\}$ form the canonical basis of $\mathbb{Z}^2$. We note an edge in $\mathcal{A}$ as $\mathfrak{e} = (x,y)$ and its transpose by $\overline{\mathfrak{e}} = (y,x)$. Given a vertex x, we denote by $\mathcal{A}_x = \{\mathfrak{e} \in \mathcal{A} : \mathfrak{e} = (x,y)\}$.
Les us denote by $X = (\mathcal{V}, \mathcal{A})$ this graph structure on $\mathbb{Z}^2$, and consider

$$C^0(X) := \{f : \mathcal{V} \to \mathbb{C}\} \; ; \qquad C^1(X) := \{f : \mathcal{A} \to \mathbb{C} \mid f(\mathfrak{e}) = -f(\overline{\mathfrak{e}})\} \; .$$

We will denote by $C(X)$ the direct sum $C^0(X) \oplus C^1(X)$.
For $f, g \in C(X)$ we take the inner product

$$\langle f, g \rangle := \sum_{x \in \mathcal{V}} f(x)\overline{g(x)} + \frac{1}{2} \sum_{\mathfrak{e} \in \mathcal{A}} f(\mathfrak{e})\overline{g(\mathfrak{e})} \; .$$

Dirac-type operator

The *difference operator* $d : C^0(X) \to C^1(X)$ is defined by:

$$df(\mathfrak{e}) := f(y) - f(x), \text{ for } \mathfrak{e} = (x, y) \ .$$

Its formal adjoint $d^* : C^1(X) \to C^0(X)$ is given by the finite sum

$$d^* f(x) = - \sum_{\mathfrak{e} \in A_x} f(\mathfrak{e}) \ .$$

Define the operator on $l^2(X)$

$$H_0 = \begin{pmatrix} m & d^* \\ d & -m \end{pmatrix} \ ,$$

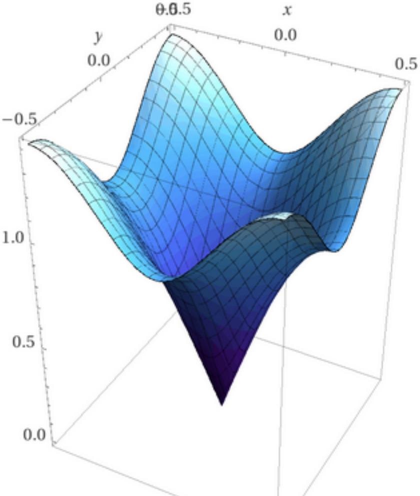
where m is just the multiplication operator by the constant $m \geq 0$.
It is easy to see that

$$H_0^2 = \begin{pmatrix} \Delta_{\mathcal{V}} + m^2 & 0 \\ 0 & \Delta_{\mathcal{A}} + m^2 \end{pmatrix} \ ,$$

so it is natural to think of H_0 as a Dirac-type operator.

H_0 is a bounded self-adjoint operator that is analytically fibered over $\mathbb{T}^2$. The analysis of its band functions permit us to show that the spectrum of H_0 is

$$\sigma(H_0) = \sigma_{ac}(H_0) = [-\sqrt{m^2+8}, -m] \cup [m, \sqrt{m^2+8}] \ .$$



There exist a discrete set $\mathcal{T}$ of thresholds in the spectrum of H_0:

$$\mathcal{T} = \left\{ \pm m, \pm\sqrt{m^2+4}, \pm\sqrt{m^2+8} \right\} \ .$$

Perturbation and the spectral shift function

We will study spectral properties of $H_0 \pm V$, where $V \geq 0$ is a potential that is defined both on vertices and edges, and satisfies $V(\mathfrak{e}) = V(\bar{\mathfrak{e}})$.
Use the notation

$$H_\pm := H_0 \pm V \ .$$

Then, its is known that when V is of trace class there exist unique functions $\boldsymbol{\xi}(\lambda; H_\pm, H_0)$ in $L^1(\mathbb{R})$ that satisfy the trace formula

$$\mathrm{Tr}(f(H_\pm) - f(H_0)) = \int_{\mathbb{R}} d\lambda \, \boldsymbol{\xi}(\lambda; H_\pm, H_0) f'(\lambda),$$

for all $f \in C_0^\infty(\mathbb{R})$. This is called the Spectral Shift Function (SSF) and is an important object in the analysis of linear operators. For instance, it is related to the scattering matrix $S(\lambda; H_\pm, H_0)$ by the Birman-Krein formula

$$\det S(\lambda; H_\pm, H_0) = e^{-2\pi i \boldsymbol{\xi}(\lambda; H_\pm, H_0)}, \quad \text{a.e. } \lambda \in \sigma_{\mathrm{ac}}(H_0).$$

It also counts the number of discrete eigenvalues of $H_\pm$ outside the essential spectrum.

Main results: bounded case

Define the following real-valued functions on $\mathbb{Z}^2$:

$$v_1(\mu) := V(\mu) \quad ; \quad v_2(\mu) := V(\mu\mathfrak{e}_1) \quad \text{and} \quad v_3(\mu) := V(\mu\mathfrak{e}_2) \ .$$

To study the most basic properties of the SSF it will suffice to assume that

$$\sum_{\mu\in\mathbb{Z}^2, \langle\mu\rangle > N} |v_j(\mu)| = O(N^{-2\beta_j}), \quad N \to \infty \ , \tag{A}$$

with $0 < \beta_j \leq 1$ for $j = 1, 2, 3$, where $\langle\mu\rangle = (1 + |\mu|^2)^{1/2}$.

Theorem

On any compact set $\mathcal{K} \subset \mathbb{R}\backslash\{-m\}$,

$$\sup_{\lambda\in\mathcal{K}} \boldsymbol{\xi}(\lambda; H_\pm, H_0) < \infty \ .$$

The study of the SSF at hyperbolic thresholds seems to have not been carried out before in any model. However, there is a growing interest in the understanding of the spectral properties of discrete Hamiltonians inside their continuous spectrum, where these kind of thresholds appear sometimes.

Main results: unbounded case

One consequence of the previous theorem is that the only possible point of unbounded growth of the SSF is $-m$.
To study this case we impose the following condition: there exist a non-zero matrix Γ such that.

$$\Gamma := \lim_{|\mu|\to\infty} |\mu|^{\gamma} \begin{pmatrix} v_2(\mu) & 0 \\ 0 & v_3(\mu) \end{pmatrix} . \tag{B}$$

In the case of existence of this matrix we define the constant

$$\mathcal{C} := \pi \int_{\mathbb{T}^2} \mathrm{Tr}\left(\left(\mathcal{A}(\xi)^* \Gamma \mathcal{A}(\xi)\right)^{2/\gamma}\right) \mathrm{d}\xi ,$$

where

$$\mathcal{A} := \begin{pmatrix} b(\xi) r(\xi)^{-1/2} & 0 \\ a(\xi) r(\xi)^{-1/2} & 0 \end{pmatrix} ,$$

and the functions a, b and $r = a^2 + b^2$ are related to the band functions of H_0.

Theorem

Suppose that v_1 satisfies (A) *with $\beta_1 > 0$, and that v_2 and v_3 satisfy* (B) *with $\gamma > 2$. Then,*

$$\boldsymbol{\xi}(\lambda; H_-, H_0) = \begin{cases} -\mathcal{C}|\lambda + m|^{-2/\gamma}(1 + o(1)) & \text{if } \lambda \uparrow -m \ , \\ O(1) & \text{if } \lambda \downarrow -m \ , \end{cases}$$

and

$$\boldsymbol{\xi}(\lambda; H_+, H_0) = \begin{cases} O(|\ln(|\lambda + m|)|) & \text{if } \lambda \uparrow -m \ , \\ \mathcal{C}|\lambda + m|^{-2/\gamma}(1 + o(1)) & \text{if } \lambda \downarrow -m \ . \end{cases}$$

Remark 1: The asymptotic order of the SSF clearly comes form V which may be interpreted as the contribution of the constant band at $-m$.
Maybe more interesting is the contribution of the non-constant band, which is encoded in the constant $\mathcal{C}$. This constant contains an explicit interaction between the perturbation and the whole non-constant band functions.

Remark 2: When $m > 0$ the interval $(-m, m)$ is a gap in the essential spectrum of $H_\pm$. Then, for $\lambda \in (-m, m)$ we can consider the function

$$\mathcal{N}^\pm(\lambda) = \mathrm{Rank}\mathbb{1}_{(\lambda,m)}(H_\pm)$$

For $\lambda \in (-m, m)$

$$\boldsymbol{\xi}(\lambda; H_\pm, H_0) = \pm\mathcal{N}^\pm(\lambda) + O(1) \ .$$

Then, the theorem provides also the asymptotic distribution of the eigenvalues of $H_\pm$.

Remark 3: The previous theorem is proved by a reduction of the SSF $\boldsymbol{\xi}$ to an eigenvalue counting function, and by using a general theorem on the spectral asymptotics for integral operators with toroidal symbol. To our best knowledge this theorem is new and may be of independent interest.

Analysis of the unperturbed operator

We introduce

$$a(\xi) := (-1 + e^{-2\pi i \xi_1}), \quad b(\xi) := (-1 + e^{-2\pi i \xi_2}) \ .$$

Proposition (Parra 2017)

The operator H_0 is unitarily equivalent, by a transformation $\mathcal{U}$, to a matrix-valued multiplication operator in $L^2(\mathbb{T}^2, \mathbb{C}^3)$ given by the real-analytic function

$$h_0(\xi) = \begin{pmatrix} m & a(\xi) & b(\xi) \\ \overline{a(\xi)} & -m & 0 \\ \overline{b(\xi)} & 0 & -m \end{pmatrix} \ .$$

For $\xi \in \mathbb{T}^2$, the characteristic polynomial associated to $h_0(\xi)$ is given by

$$p_\xi(z) = (m+z)\Big((m^2 - z^2) + (|a(\xi)|^2 + |b(\xi)|^2)\Big) \ .$$

Then there are three band functions:

$$z_0(\xi) = -m \ , \quad z_\pm(\xi) = \pm\sqrt{m^2 + |a(\xi)|^2 + |b(\xi)|^2} \ .$$

We set $r_m(\xi) := m^2 + |a(\xi)|^2 + |b(\xi)|^2$. To compute the thresholds notice that

$$\nabla z_\pm(\xi) = \pm \frac{2\pi}{r_m(\xi)^{\frac{1}{2}}} (\sin(2\pi\xi_1), \sin(2\pi\xi_2)).$$

Ideas of the proof: SSF representation

Define for $z \in \mathbb{C}^+$ the operator $K(z) = V^{1/2}(H_0 - z)^{-1}V^{1/2}$.
The norm limits

$$\lim_{\delta \downarrow 0} K(\lambda + i\delta) =: K(\lambda + i0) \tag{C}$$

exist for any $\lambda \in \mathbb{R} \setminus \mathcal{T}$.
We have the Pushnitski representation:

$$\boldsymbol{\xi}(\lambda; H_\pm, H_0) = \pm\frac{1}{\pi}\int_{\mathbb{R}} \frac{dt}{1+t^2} n_\mp(1; \operatorname{Re} K(\lambda + i0) + t \operatorname{Im} K(\lambda + i0)),$$

where $n_\pm(1;\cdot)$ is the function than counts the number of eigenvalues bigger (smaller) than 1 (-1).

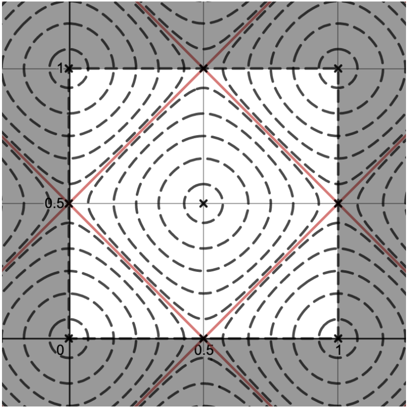
Case $m = 0$

We can see that $V^{1/2}(H_0 - z)^{-1}V^{1/2}$ is unitary equivalent to

$$\frac{V}{z}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + V^{1/2}\mathcal{U}^* \frac{1}{z(r_0(\xi) - z^2)} \begin{pmatrix} z^2 & za(\cdot) & zb(\cdot) \\ z\overline{a(\cdot)} & |a(\cdot)|^2 & -\overline{a(\cdot)}b(\cdot) \\ z\overline{b(\cdot)} & -\overline{b(\cdot)}a(\cdot) & |b(\cdot)|^2 \end{pmatrix} \mathcal{U}V^{1/2}\ .$$

The last term can be written in the form

$$\int_0^{\sqrt{8}} \frac{du}{u^2 - z^2} \int_{r_0^{-1}(u^2)} d\gamma \frac{T_z(\gamma)}{|\nabla r_0(\gamma)|}$$



Proposition (Limiting absorption principle)

Let $\lambda \in \mathbb{R} \backslash \mathcal{T}$ *and set* $z = \lambda + i\delta$. *Then as* $\delta \downarrow 0$, *in the* $\mathfrak{S}_1$*-norm*

$$\int_0^{\sqrt{8}} \frac{d\rho}{u^2 - z^2} \int_{r_0^{-1}(u^2)} d\gamma \frac{T_z(\xi)}{|\nabla r_0(\xi)|} \longrightarrow p.v. \int_0^{\sqrt{8}} du \frac{2u}{(u^2 - \lambda^2)} \int_{r_0^{-1}(u^2)} d\gamma \frac{T_\lambda(\xi)}{|\nabla r_0(\xi)|} - i\pi \int_{r_0^{-1}(\lambda^2)} d\gamma \frac{T_\lambda(\xi)}{|\nabla r_0(\xi)|}$$

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Proposition (Effective Hamiltonian)

$$\mathcal{Q} = V^{1/2} \mathcal{U}^* \frac{1}{r_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -|b|^2 & -\overline{a}b \\ 0 & -\overline{b}a & -|a|^2 \end{pmatrix} \mathcal{U} V^{1/2}$$

i.e.,

$$\boldsymbol{\xi}(\lambda; H_\pm, H_0) \sim \pm n_\mp(\lambda; \mathcal{Q}), \quad \lambda \downarrow 0.$$

We can write $\mathcal{Q} = V^{1/2}\mathcal{U}^* \frac{1}{r} \begin{pmatrix} -|b|^2 & -\overline{a}b \\ -\overline{b}a & -|a|^2 \end{pmatrix} \mathcal{U} V^{1/2}$, acting in $l^2(X)$.

Then, if we define

$$q = V^{1/2}\mathcal{U}^* \frac{1}{\sqrt{r}} \begin{pmatrix} b & -\overline{a} \\ a & \overline{b} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have that $\mathcal{Q} = -qq^*$, and for all $s > 0$

$$n_{\mp}(s; \mathcal{Q}) = n_{\pm}(s; q^* q).$$

The operator q^*q is an integral operator in $L^2(\mathbb{T}^d)$ with kernel

$$\frac{\overline{b(\xi)}}{\sqrt{r(\xi)}} \widehat{v_2}(\xi - \eta) \frac{b(\eta)}{\sqrt{r(\eta)}} + \frac{\overline{a(\xi)}}{\sqrt{r(\xi)}} \widehat{v_3}(\xi - \eta) \frac{a(\eta)}{\sqrt{r(\eta)}}$$

For $B \in L^2(\mathbb{T}^d)$, let Ψ be the integral operator in $L^2(\mathbb{T}^d)$ with kernel

$$B(\xi)\widehat{v}(\xi-\eta)\overline{B}(\eta) \ .$$

Let us consider the class of Symbols $S^\gamma(\mathbb{Z}^d)$ given by the functions $v : \mathbb{Z}^d \to \mathbb{C}$ that satisfies for any multi-index α

$$|\mathrm{D}^\alpha v(\mu)| \leq C_\alpha \langle\mu\rangle^{-\gamma-|\alpha|},$$

where $\mathrm{D_j} v(\mu) := v(\mu+\delta_j) - v(\mu)$, and $\mathrm{D}^\alpha := \mathrm{D}_1^{\alpha_1} ... \mathrm{D}_d^{\alpha_d}$.
For $\gamma > d$, assume $\{v\}_{k=1}^N \in S^\gamma(\mathbb{Z}^d)$, and with v viewed as a multiplication operator

$$n_\pm(\lambda; v)(\mu) \sim c_\pm \lambda^{-d/\gamma}, \quad \lambda \downarrow 0.$$

Theorem

$$n_\pm(\lambda; \Psi) \sim \mathcal{C}_{B\pm} \lambda^{-d/\gamma}, \quad \lambda \downarrow 0,$$

where

$$\mathcal{C}_{B\pm} = c_\pm \int_{\mathbb{T}^d} d\xi \left(|B(\xi)|^2\right)^{d/\gamma} \ .$$

Thank you for your attention!