Long-range scattering for a homogeneous type nonlinear Schrödinger equation

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Introduction

We consider the nonlinear Schrödinger equation

(NLS) $$i\partial_t u - H(t)u = F(u),\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where $u = u(t,x) \in \mathbb{C}$, $d \le 3$,

$$H(t) = -\frac{\Delta}{2} + \sigma(t)\frac{|x|^2}{2}, \quad \sigma(t) \in L^\infty([0,\infty)\,;\,\mathbb{R}) \quad (\text{e.g., } \sigma(t) \sim t^{-2},\ t \gg 1).$$

F: A general homogeneous of degree $1 + \frac{2}{d(1-\lambda)}$; $\qquad p_c := \frac{2}{d(1-\lambda)}$

(H) $$F(\alpha z) = \alpha^{1+p_c} F(z), \quad \lambda \in [0, 1/2) \quad (\alpha > 0,\ z \in \mathbb{C}).$$

Examples of the nonlinearity

- Gauge-invariant case: $F(u) = |u|^{p_c} u$, i.e., $F(e^{i\theta} z) = e^{i\theta} F(z)$.
- Non Gauge-invariant case: $F(u) = |u|^{p_c - 1} u^2$ $(d \le 2)$, $|\mathrm{Re}\, u|^{p_c}\, \mathrm{Re}\, u$

Our main subject

We investigate the final state problem for (NLS).

"For given u_p, find $u(t) \to u_p(t)$ as $t \to \infty$."

Critical exponent $p_c = 2/d$ in the case $\sigma(t) \equiv 0$

$$(\text{NLS}_0) \qquad i\partial_t u + \frac{1}{2}\Delta u = |u|^p u, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

(NLS$_0$) & $u(t) \approx e^{it\Delta/2}u_+$ as $t \to \infty$

$$\Leftrightarrow \; u(t) = e^{it\Delta/2}u_+ - i\int_t^\infty e^{i(t-s)\Delta/2}|u(s)|^p u(s)ds$$

with given suitable u_+ called "final state".

- Dispersive estimate: $\left\|e^{it\Delta/2}u_+\right\|_{L^\infty} \leqslant Ct^{-\frac{d}{2}} \|u_+\|_{L^1}$.
- Isometric property: $\left\|e^{it\Delta/2}u_+\right\|_{L^2} = \|u_+\|_{L^2}$.

If $u(t) \approx e^{it\Delta/2}u_+$, then

$$\left\|\int_t^\infty e^{i(t-s)\Delta/2}|e^{is\Delta/2}\phi_+|^p e^{is\Delta/2}\phi_+ \, ds\right\|_{L^2} \leqslant \|\phi_+\|_{L^1}\|\phi_+\|_{L^2}\int_t^\infty s^{-\frac{d}{2}p}\,ds.$$

It is expected that $u(t) \to e^{it\Delta/2}u_+$ when $-\frac{d}{2}p < -1 \Leftrightarrow p > \frac{2}{d}$.

Previous works (Gauge invariant case)

(NLS$_0$) $$i\partial_t u + \frac{1}{2}\Delta u = |u|^p u, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

If $p > 2/d$, then $u(t) \to \underbrace{(it)^{-\frac{d}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{x}{t}\right)}_{\approx e^{it\Delta/2}u_+}$ as $t \to \infty$ (u_+: final state).

Short-range scattering cf. Tsutsumi-Yajima '84.

If $p \leqslant 2/d$, then $\nexists u(t) \to e^{it\Delta/2}u_+$ as $t \to \infty$.

Failure of scattering cf. Barab '83.

If $p = 2/d$, then

$$u(t) \to (it)^{-\frac{d}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{x}{t}\right) \exp\left(-i\left|\widehat{u_+}\left(\frac{x}{t}\right)\right|^{\frac{2}{d}} \log t\right) \text{ as } t \to \infty.$$

Long-range scattering cf. Ozawa '91, $d = 1$, Ginibre-Ozawa '93, $d = 2, 3$.
Hayashi-Naumkin '06.

Asymptotic profile with a logarithmic phase correction

We follow Ozawa (1991) and Hayashi-Naumkin (2006).

Step 1. Find an appropriate asymptotic behavior of solutions: $U_0(t) := e^{it\Delta/2}$

- (NLS$_0$) with $F(u) = |u|^{p_c}u \to i\partial_t(\mathcal{F}U_0(-t)u) = \mathcal{F}U_0(-t)(|u|^{p_c}u)$.
- $w(t) := \widehat{u_+}\exp\left(-i|\widehat{u_+}|^{p_c}\log t\right)$ solves $i\partial_t w = \dfrac{1}{t}|w|^{p_c}w$.

Key lemma (Dollard decomposition)

$U_0(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t)$, where $\mathcal{M}(t) = e^{\frac{i|x|^2}{2t}}$, $[\mathcal{D}(t)f](x) = (it)^{-\frac{d}{2}}f(x/t)$.

- Set $u_p(t) = \mathcal{M}(t)\mathcal{D}(t)w(t)$.

  NOTE: $U_0(t)\mathcal{F}^{-1} = \mathcal{M}(t)\mathcal{D}(t) + \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}(\underbrace{\mathcal{M}(t)-1}_{\lesssim |x|^{2\theta}|t|^{-\theta},\ \theta\in[0,1]})\mathcal{F}^{-1}$.

Step 2. By the contraction argument, show $\exists u(t) \to u_p$ as $t\to\infty$.

$$\because)\quad i\partial_t\left(\mathcal{F}U_0(-t)u - w\right) = \mathcal{F}U_0(-t)\left(|u|^{p_c}u - |u_p|^{p_c}u_p\right) + \mathcal{F}\mathcal{M}^{-1}(t)\mathcal{F}^{-1}\frac{1}{t}|w|^{p_c}w.$$

Previous works (Non Gauge-invariant case)

Assume that the nonlinearity is described as

$$F(u) = \underbrace{g_0|u|^{\frac{2}{d}+1}}_{\text{Non-oscillating part}} + \underbrace{g_1|u|^{\frac{2}{d}}u}_{\text{Resonant part}} + \underbrace{\sum_{n\neq 0,1} g_n|u|^{1+\frac{2}{d}-n}u^n}_{\text{Non-resonant part}}.$$

Roughly speaking, we have

- If $g_0 = 0$ and $g_1 \in \mathbb{R} \setminus \{0\}$, Long-range scattering
- If $g_0 = 0$ and $g_1 = 0$, Short-range scattering
- If $g_0 \neq 0$, Failure of scattering & $\exists$ blow-up sol. in finite time

- Non-resonant part is a finite sum:

  Moriyama-Tonegawa-Tsutsumi'04, Hayashi-Naumkin-Wang'11, Shimomura-Tsutsumi'06, Ikeda-Wakasugi'13, etc...
- Non-resonant part is an infinite sum:

  Masaki-M.'18, '19, Masaki-M.-Uriya'19

The case $\sigma(t) \not\equiv 0$

Consider the initial value problem for an ordinary differential equation

$$(1)\quad \begin{cases} \zeta_j''(t) + \sigma(t)\zeta_j(t) = 0, \quad \sigma(t) = \begin{cases} \sigma_0 & (t \in [0, r_0)), \\ \sigma_1 t^{-2} & (t \in [r_0, \infty)), \end{cases} \quad j = 1,\ 2, \\ \zeta_1(0) = 1,\ \zeta_1'(0) = 0,\ \zeta_2(0) = 0,\ \zeta_2'(0) = 1, \end{cases}$$

where $\sigma_0 \in [0, 1/4)$, σ_1 and r_0 are suitable constants (s.t. $\exists\ C^2$-sol. to (1)).
For any $t \geq r_0$, then

$$y''(t) + \sigma(t)y(t) = 0, \quad \lambda = (1 - \sqrt{1 - 4\sigma_0})/2$$

have two solutions t^λ and $t^{1-\lambda}$.

Then $\exists \zeta_j \in C^2$: sol. to (1) described by

$$\zeta_1(t) = \begin{cases} \cos(\omega_0 t) & (t \in [0, r_0)), \\ c_1 t^\lambda & (t \in [r_0, \infty)), \end{cases} \qquad \zeta_2(t) = \begin{cases} \frac{1}{\omega} \sin(\omega_0 t) & (t \in [0, r_0)), \\ c_2 t^{1-\lambda} + c_3 t^\lambda & (t \in [r_0, \infty)), \end{cases}$$

where $\omega_0 = \sqrt{\sigma_0}$ and $c_i > 0$.

Lemma 1 (Korotyaev'90, Kawamoto-Muramatsu'21)

$U(t) = \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_2'(t)}\right)\mathcal{D}(\zeta_2(t))\mathcal{F}\mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_1(t)}\right)$, where $U(t)$ is a propagator of $H(t)$.

Remark 1

We can handle $\sigma(t)$ in a more general assumption.

- Dispersive estimate: $\|U(t)\phi\|_{L^\infty} \leqslant C|\zeta_2(t)|^{-\frac{d}{2}}\|\phi\|_{L^1} = O(t^{-\frac{d}{2}(1-\lambda)})$.

  The fact causes the change of p_c from $2/d$ to $2/d(1-\lambda)$.
- Isometric property: $\|U(t)\phi\|_{L^2} = \|\phi\|_{L^2}$.
- $|\mathcal{M}(\zeta_2(t)/\zeta_1(t)) - 1| \leqslant C|x|^{2\theta}|\zeta_1(t)/\zeta_2(t)|^{-\theta}, \quad \theta \in [0,1]$.

  NOTE: $\zeta_1(t)/\zeta_2(t) = O(t^{-(1-2\lambda)})$ if $\zeta_1(t) = O(t^\lambda)$ and $\zeta_2(t) = O(t^{1-\lambda})$.

If $F(u) = |u|^{p_c}u$, Kawamoto '21 proves that (NLS) admits a solution such that

$$u(t) \to \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_2'(t)}\right)\mathcal{D}(\zeta_2(t))\,\widehat{u_+}\exp\left(-i|c_2|^{-\frac{1}{1-\lambda}}|\widehat{u_+}|^{p_c}\log t\right) \text{ as } t \to \infty.$$

Our aim here is to extend the above result into the non gauge-invariant case.

Decomposition of general homogeneous nonlinearity

Recalling the condition (H) (i.e., $F(\alpha z) = \alpha^{1+p_c} F(z)$),

$$\begin{aligned} F(u) &= |u|^{1+p_c} F\left(\frac{u}{|u|}\right) \\ &= |u|^{1+p_c} F\left(e^{i\theta}\right) \quad (\theta := \arg u) \\ &= |u|^{1+p_c} g(\theta) \quad (g(\theta) := F\left(e^{i\theta}\right)). \end{aligned}$$

NOTE: F : homogeneous $\sim g$: 2π-periodic.

The Fourier series expansion

$$g(\theta) = \sum_{n\in\mathbb{Z}} g_n e^{in\theta}, \quad g_n := \frac{1}{2\pi}\int_0^{2\pi} g(\theta) e^{-in\theta} d\theta$$

gives us

$$F(u) = |u|^{1+p_c} \sum_{n\in\mathbb{Z}} g_n \left(\frac{u}{|u|}\right)^n = \sum_{n\in\mathbb{Z}} g_n |u|^{1+p_c-n} u^n.$$

By the procedure, we can decompose $F(u)$ into

$$F(u) = \underbrace{g_0|u|^{p_c+1}}_{\text{Non-oscillating part}} + \underbrace{g_1|u|^{p_c}u}_{\text{Resonant part}} + \underbrace{\sum_{n\neq 0,1} g_n|u|^{1+p_c-n}u^n}_{\text{Non-resonant part}}.$$

cf. Sunagawa'06, Masaki-Segata'16. MM'18, '19, MMU'19.

Assumption

(i) $\lambda = \lambda_d \geq 0$ satisfies $\lambda_1 < 6 - \sqrt{33}$, $\lambda_2 < \frac{1}{5}$, $\lambda_3 < \frac{13 - 2\sqrt{37}}{21}$.

(ii) F satisfies (H) such that a corresponding 2π-periodic function $g(\theta)$ satisfies

$$g_0 = 0, \quad g_1 \in \mathbb{R}, \quad \sum_{n\in\mathbb{Z}} |n|^{1+a(\lambda)+\eta}|g_n| < \infty$$

for some $\eta > 0$, where $a(\lambda) = a_d(\lambda)$ is a certain monotone increasing function. Note that $a(\lambda) \in (0, 3/4)$ if $d = 1$, $a(\lambda) \in (0, 1/2)$ if $d = 2$, and $a(\lambda) \in \left(0, (2\sqrt{37} - 11)/2\right)$ if $d = 3$. In particular, g is Lipschitz continuous.

Notations: For any m, $s \in \mathbb{R}$, $\quad (\langle x\rangle := (1 + |x|^2)^{1/2}, x \in \mathbb{R})$

$$H^{0,m} := \{f : \mathbb{R}^d \to \mathbb{C}\,; \langle x\rangle^m f \in L^2\},\ \dot{H}^s := \{f : \mathbb{R}^d \to \mathbb{C}\,; |\nabla|^s f \in L^2\}.$$

Main result

Theorem 1 (Kawamoto-M.)

Let $\lambda \geqslant 0$ satisfy Assumption (i). Suppose that F satisfies Assumption (ii) for some $\eta > 0$. Set $\delta = \delta_d > 0$ such that

$$(2) \qquad \frac{1+4\lambda-\lambda^2}{2(1-2\lambda)} < \delta_1 < 2, \quad \frac{d(\lambda+1)}{2(1-2\lambda)} < \delta_d < \min(2, 1+p_c) \quad (d=2,\,3)$$

with $\delta - d/2 < 2(a(\lambda)+\eta)$. Take $\delta' = \max(1,\delta)$ if $d=1$ and $\delta' = \delta$ if $d = 2,\,3$. Then, $\forall u_+ \in H^{0,\delta'} \cap \dot{H}^{-\delta}$ with $\|\widehat{u_+}\|_\infty \ll 1$, $\exists T \geq r_0$ and $\exists! u \in C([T,\infty)\,;\, L^2)$: sol. to (NLS) which satisfies

$$\sup_{t\geq T} \tau^{b-\lambda}\|u(t)-u_p(t)\|_{L^2} + \sup_{t\geq T} t^{b-2\lambda} \left\| \langle\cdot\rangle^{-\lambda/q} (u-u_p)(\cdot) \right\|_{L^q([t,\infty),L^r)} < \infty$$

for any $b \in (2\lambda, \lambda + \delta(1-2\lambda)/2)$, where

$$u_p(t) = \mathcal{M}\left(\zeta_2(t)/\zeta_2'(t)\right)\mathcal{D}\left(\zeta_2(t)\right)\widehat{u_+}\exp\left(-ig_1|c_2|^{-\frac{1}{1-\lambda}}\,|\widehat{u_+}|^{\frac{2}{d(1-\lambda)}}\log t\right),$$

and $(q,r,d) = (4,\infty,1)$, $\left(\frac{2}{1-2\varepsilon_1}, \frac{1}{\varepsilon_1}, 2\right)$, $\left(\frac{2}{1-2\varepsilon_1}, \frac{6}{1+4\varepsilon_1}, 3\right)$ with $0 < \varepsilon_1 \ll 1$.

Remarks on the main result

- $F(u) = |\operatorname{Re} u|^{p_c} \operatorname{Re} u$ satisfies Assumption (ii). Long-range scattering
  The corresponding periodic function:

$$g(\theta) = |\cos\theta|^{p_c} \cos\theta,$$

$$g_n = \begin{cases} \dfrac{\Gamma\left(\frac{p_c+3}{2}\right)\Gamma\left(\frac{n-1-p_c}{2}\right)}{\sqrt{\pi}\Gamma(-\frac{p_c}{2})\Gamma(\frac{n+3+p_c}{2})} = O(|n|^{-p_c-2}) & n\text{: odd,} \\ 0 & n\text{: even.} \end{cases}$$

- $F(u) = |\operatorname{Re} u|^{p_c} \operatorname{Re} u - i|\operatorname{Im} u|^{p_c} \operatorname{Im} u$ satisfies Assumption (ii) and $g_1 = 0$.
  Short-range scattering
  The corresponding periodic function:

$$g(\theta) = |\cos\theta|^{p_c} \cos\theta - i|\sin\theta|^{p_c} \sin\theta,$$

$$g_n = \begin{cases} \dfrac{-2\Gamma\left(\frac{p_c+3}{2}\right)\Gamma\left(\frac{n-1-p_c}{2}\right)}{\sqrt{\pi}\Gamma(-\frac{p_c}{2})\Gamma(\frac{n+3+p_c}{2})} = O(|n|^{-p_c-2}) & n \equiv 3 \mod 4, \\ 0 & \text{otherwise.} \end{cases}$$

NOTE: The Stirling formula $\Gamma(z) \sim \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z$ as $|z| \to \infty$.

Remark 2

We need the assumption (i) to take δ in (2). In fact,

- $d = 1$: (2) $\to \lambda^2 - 12\lambda + 3 > 0$ solved by $\lambda < 6 - \sqrt{33} \sim 0.255$.
- $d = 2$: (2) $\to \lambda < 1/5$.
- $d = 3$: (2) $\to 21\lambda^2 - 26\lambda + 1 > 0$ solved by $\lambda < (13 - 2\sqrt{37})/21 \sim 0.0397$.

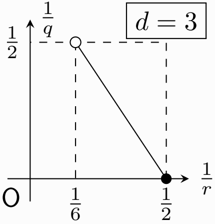
The assumption (i) is weaker than that in Kawamoto'21. This implies an improvement of the condition for δ. To relax the condition for λ, we need to choose the admissible pair near the end-point.

NOTE: End-point Strichartz' estimate for $U(t)$ in $d = 2, 3$ can not be expected: $\quad I := [t, \infty)$

$$\left\| \langle \cdot \rangle^{-\lambda/q} u \right\|_{L^q(I\,;\,L^r)} \lesssim \|u_+\|_{L^2} + \left\| \langle \cdot \rangle^{\lambda/q} F(u) \right\|_{L^{\tilde{q}'}(I\,;\,L^{\tilde{r}'})},$$

(q, r), $(\tilde{q}, \tilde{r})$: admissible pair $\overset{\text{def}}{\Leftrightarrow}$ $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$, $q \in (2, \infty]$.

cf. Kawamoto-Yoneyama'18, Kawamoto'20.



Remark 3

When $\lambda = 0$, we improve the regularity condition of u_+ compared with MM'18 & MMU'19. Other conditions in Theorem 1 coincide those in previous works.

Sketch of the proof of Theorem 1

We follow HNW'11, MMU'19 and Kawamoto'21. Set

$$F(u) = g_1|u|^{\frac{2}{d}}u + \sum_{n\neq 0,1} g_n|u|^{\frac{2}{d}-n}u^n =: \mathcal{G}_d(u) + \mathcal{N}_d(u), \quad c_+ = |c_2|^{1/(1-\lambda)},$$

$$R(t) = \mathcal{M}\left(\zeta_2(t)/\zeta_2'(t)\right)\mathcal{D}(\zeta_2(t))\left(\mathcal{F}\mathcal{M}\left(\zeta_2(t)/\zeta_1(t)\right)\mathcal{F}^{-1} - 1\right).$$

(NLS) & $u(t) \to U(t)u_+$ as $t \to \infty \Leftrightarrow$

$$u(t) - u_p(t) = \underbrace{i\int_t^\infty U(t-s)\left(F(u(s)) - F(u_p(s))\right)ds}_{\substack{\text{Strichartz' estimate.} \to \text{ To prove the existence of sol.,} \\ \text{we require } b > \exists b_d(\lambda) > \frac{d}{4}, \text{ by which (2) for } \delta \text{ is determined.}}} + \mathcal{A}(t) + \mathcal{E}_{\mathrm{r}}(t) + \mathcal{E}_{\mathrm{nr}}(t),$$

where

$$\mathcal{A}(t) = i\int_t^\infty U(-t)\mathcal{F}^{-1}\left(\frac{c_+s}{\zeta_2(s)^{1/(1-\lambda)}} - 1\right)F(\widehat{w}(s))\frac{ds}{c_+s},$$

$$\mathcal{E}_{\mathrm{r}}(t) = R(t)\widehat{w}(t) - i\int_t^\infty U(t-s)R(s)\mathcal{G}_d(\widehat{w}(s))\frac{ds}{\zeta_2(s)^{1/(1-\lambda)}},$$

$$\mathcal{E}_{\mathrm{nr}}(t) = i\int_t^\infty U(t-s)\mathcal{N}(u_p(s))\,ds.$$

$$R(t) = \mathcal{M}\left(\zeta_2(t)/\zeta_2'(t)\right)\mathcal{D}(\zeta_2(t))\underbrace{\left(\mathcal{F}\mathcal{M}\left(\zeta_2(t)/\zeta_1(t)\right)\mathcal{F}^{-1} - 1\right)}_{\lesssim \left||\nabla|^2 t^{-(1-2\lambda)}\right|^{\delta/2}}.$$

$$\mathcal{A}(t) = i\int_t^\infty U(t)\mathcal{F}^{-1}\underbrace{\left(\frac{c_+ s}{\zeta_2(s)^{1/(1-\lambda)}} - 1\right)}_{\lesssim s^{\frac{\lambda}{1-\lambda}-1}} F(\widehat{w}(s))\frac{ds}{c_+ s},$$ harmless

$$\mathcal{E}_{\mathrm{r}}(t) = R(t)\widehat{w}(t) - i\int_t^\infty U(t-s)R(s)\mathcal{G}_d(\widehat{w}(s))\frac{ds}{\zeta_2(s)^{1/(1-\lambda)}},$$ harmless

$$\mathcal{E}_{\mathrm{nr}}(t) = \underbrace{i\int_t^\infty U(t-s)\mathcal{N}(u_p(s))\,ds.}_{\text{Integration by part for } t}$$

$+$

Time dependent regularizing op. $\mathcal{K}_\psi := \psi\left(\frac{-i\nabla}{|n|t^{1/2}}\right)$ by [HNW'11], [MM'18], [MMU'19]

$$\underset{\text{modification}}{\longrightarrow} \mathcal{K}_\psi := \psi\left(\frac{-i\nabla}{|n|t^{(1-2\lambda)/2}}\right) \quad (\psi \in \mathscr{S} \text{ with } \nabla\psi(0) = 0).$$

$a(\lambda)$ in Assumption (ii) appears from the restriction of δ depending on λ.

Thank you for your attention.