Energy decay for small solutions to semilinear wave equations with weakly dissipative structure

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Introduction

We consider the Cauchy problem

$$\text{(NLW)} \quad \begin{cases} \Box u = F(\partial u), & t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = \varepsilon f(x), & \\ \partial_t u(0,x) = \varepsilon g(x), & x \in \mathbb{R}^2. \end{cases}$$

- $\Box = \partial_t^2 - \Delta = \partial_0^2 - (\partial_1^2 + \partial_2^2)$, $\quad \partial u = (\partial_0 u, \partial_1 u, \partial_2 u)$,
  $\partial_0 = \partial_t = \partial/\partial t$, $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$.
- $u = u(t,x)$: $\mathbb{R}$-valued unknown.
- $F(\partial u)$: polynomial w.r.t. ∂u.
- $f, g \in C_0^\infty(\mathbb{R}^2)$, $0 < \varepsilon \ll 1$.

- $F \equiv 0$ (free wave equation $\Box u = 0$)

$$|\partial u(t,x)| \leq C(1+t)^{-1/2}, \qquad t \geq 0,\ x \in \mathbb{R}^2.$$

$$\|u(t)\|_E = \|u(0)\|_E, \qquad t \geq 0.$$

where $\|u(t)\|_E^2 = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} |\partial u(t,x)|^2 dx$.

- $F(\partial u) = O(|\partial u|^p)$ near $\partial u = 0$, $\quad p > 1$

$p > 3 \Longrightarrow$ Small Data Global Existence (SDGE) holds.
$\overset{def}{\Longleftrightarrow}$ ${}^{\forall} f, g \in C_0^\infty(\mathbb{R}^2)$, ${}^{\exists}\varepsilon_0 > 0$ s.t. $0 < \varepsilon \leq \varepsilon_0$
$\Rightarrow$ ${}^{\exists !}u$: global classical sol. to (NLW).

$p = 3$: one of the critical cases.

- $F(\partial u) = -(\partial_t u)^3$ (nonlinear damping) $\longrightarrow$ SDGE
- $F(\partial u) = (\partial_t u)^3$ $\longrightarrow$ $^\exists$blow-up solution
- $F(\partial u) = (\partial_t u)^3 - (\partial_t u)|\nabla_x u|^2$ $\longrightarrow$ SDGE

We are interested in

- ◦ structural conditions of F for SDGE,
- ◦ asymptotic behavior of the global solution u.

We put $F(\partial u) = F_{\mathrm{q}}(\partial u) + F_{\mathrm{c}}(\partial u) + F_h(\partial u)$, where

- $F_q(\partial u) = \displaystyle\sum_{j,k=0}^{2} B_{jk}(\partial_j u)(\partial_k u)$
- $F_c(\partial u) = \displaystyle\sum_{j,k,l=0}^{2} C_{jkl}(\partial_j u)(\partial_k u)(\partial_l u)$
- $F_h(\partial u) = O(|\partial u|^4)$ near $\partial u = 0$

for $B_{jk}, C_{jkl} \in \mathbb{R}$.

Remark

- $F_q = F_c \equiv 0 \Longrightarrow$ SDGE. $\rightsquigarrow$ We consider F_q and F_c.

Null condition

$$F_q(\partial u) = \sum_{j,k=0}^{2} B_{jk}(\partial_j u)(\partial_k u), \quad F_c(\partial u) = \sum_{j,k,l=0}^{2} C_{jkl}(\partial_j u)(\partial_k u)(\partial_l u)$$

Null condition

- Quadratic null condition (**QN**)

$$\overset{def}{\Longleftrightarrow} F_q(\hat{\omega}) = \sum_{j,k=0}^{2} B_{jk}\omega_j\omega_k = 0 \text{ for all } \omega \in \mathbb{S}^1.$$

- Cubic null condition (**CN**)

$$\overset{def}{\Longleftrightarrow} F_c(\hat{\omega}) = \sum_{j,k,l=0}^{2} C_{jkl}\omega_j\omega_k\omega_l = 0 \quad \text{for all } \omega \in \mathbb{S}^1.$$

Here $\hat{\omega} = (\omega_0, \omega_1, \omega_2) = (-1, \omega)$, $\omega \in \mathbb{S}^1$.

Godin(1993) cf. Klainerman(1986), Christodoulou(1986)

$$(\textbf{QN}) \text{ and } (\textbf{CN}) \quad \Longrightarrow \quad \text{SDGE for (NLW).}$$

Remarks

- Under (**QN**) and (**CN**), the solution u is asymptotically free
  - Asymptotically free $\overset{def}{\Longleftrightarrow}$ There exists $u^+ = u^+(t,x)$ satisfying $\Box u^+ = 0$ and $(u^+, \partial_t u^+)\,|_{t=0} \in \dot{H}^1 \times L^2$ such that $\lim_{t\to\infty} \|u(t) - u^+(t)\|_E = 0$.
- Moreover, $u(t)$ behaves like a non-trivial free solution as $t \to \infty$ unless $f = g \equiv 0$. In particular, $\lim_{t\to\infty} \|u(t)\|_E \neq 0$.

Agemi condition

In what follows, we always assume **(QN)**.
Let $P(\omega) = F_c(\hat{\omega})$.

$$\partial_a u \longleftrightarrow \omega_a$$
$$\omega_0 = -1,\ \omega \in \mathbb{S}^1$$

- $F_c(\partial u) = -(\partial_t u)^3 \quad \longrightarrow \quad P(\omega) \equiv 1$
- $F_c(\partial u) = (\partial_t u)^3 \quad \longrightarrow \quad P(\omega) \equiv -1$
- $F_c(\partial u) = (\partial_t u)^3 - (\partial_t u)|\nabla_x u|^2 \quad \longrightarrow \quad P(\omega) \equiv 0$

Agemi condition

- **(A)** $\overset{def}{\Longleftrightarrow}$ $P(\omega) \geq 0$ for all $\omega \in \mathbb{S}^1$.
- $(\mathbf{A}_+)$ $\overset{def}{\Longleftrightarrow}$ $P(\omega) > 0$ for all $\omega \in \mathbb{S}^1$.

$$P(\omega) = \sum_{j,k,l=0}^{2} C_{jkl}\omega_j\omega_k\omega_l \quad \textbf{(A)}\ P(\omega) \geq 0 \quad \textbf{(A}_+\textbf{)}\ P(\omega) > 0$$

Hoshiga(2008), Kubo(2007)

(A) $\Longrightarrow$ SDGE for (NLW).

Katayama-Murotani-Sunagawa(2012)

$$\textbf{(A}_+\textbf{)} \implies \|u(t)\|_E \leq \frac{C\varepsilon}{(1+\varepsilon^2\log(t+2))^{1/4-\delta}},$$

as $t \to +\infty$, where $0 < \delta \ll 1$.

Remark

- **(A)** $\iff P(\omega) \geq 0$ for all $\omega \in \mathbb{S}^1 \implies$ SDGE holds.
- **(CN)** $\iff P(\omega) = 0$ for all $\omega \in \mathbb{S}^1$

$$\rightsquigarrow \|u(t)\|_E = O(1) \text{ as } t \to +\infty$$

- **(A$_+$)** $\iff P(\omega) > 0$ for all $\omega \in \mathbb{S}^1$

$$\rightsquigarrow \|u(t)\|_E = O((\log t)^{-1/4+\delta}) \text{ as } t \to +\infty$$

Question

Does the energy decay occur when **(A)** is satisfied but **(CN)** and **(A$_+$)** are violated?

Main result

Theorem (N.-Sunagawa-Terashita JMSJ 2021)

Assume that (**QN**) and (**A**) are satisfied but (**CN**) is violated. For the global solution u to (NLW), there exist positive constants C and λ such that

$$\|u(t)\|_E \leq \frac{C\varepsilon}{(1+\varepsilon^2 \log(t+2))^\lambda}$$

for $t \geq 0$, provided that $\varepsilon > 0$ is sufficiently small.

The energy decay occurs when (**A**) is satisfied but (**CN**) is violated.

Examples

The cubic nonlinear terms below are examples of $F_{\mathrm{c}}(\partial u)$ which satisfy **(A)** but fail to satisfy **(CN)**: $\partial_a u \longleftrightarrow \omega_a$

	$F_c(\partial u)$	$P(\omega)$	λ
(A)	$-(\partial_1 u)^2 \partial_t u$	ω_1^2	$1/4-\delta$
(A)	$-(\partial_1 u)^2(\partial_t u + \partial_2 u)$	$\omega_1^2(1-\omega_2)$	$1/8-\delta$
(A)	$-(\partial_t u + \partial_2 u)^3$	$(1-\omega_2)^3$	$1/12-\delta$
($\mathbf{A}_+$)	$-(\partial_t u)^3$	1	$1/4-\delta$

$$\|u(t)\|_E \le \frac{C\varepsilon}{(1+\varepsilon^2\log(t+2))^\lambda} \qquad \delta > 0 : \text{small}$$

Outline of the proof

Lemma 1 (Key estimate)

Let $0 < \mu < 1/10$. Assume that (**QN**) and (**A**) are satisfied. If $\varepsilon > 0$ is suitably small, there exists a positive constant C, not depending on ε, such that the solution u to (NLW) satisfies

$$|\partial u(t, r\omega)| \leq \frac{C\varepsilon}{\sqrt{t}} \min\left\{ \frac{1}{\sqrt{P(\omega)\varepsilon^2 \log t}}, \frac{1}{\langle t-r\rangle^{1-\mu}} \right\}$$

for $(t, r, \omega) \in [2, \infty) \times [0, \infty) \times \mathbb{S}^1$, where $r = |x|$, $\omega = x/|x|$ and $\langle z\rangle = (1 + |z|^2)^{1/2}$.

Proof of Key estimate

From Katayama-Murotani-Sunagawa, we already know

$$|\partial u(t, r\omega)| \leq \frac{C\varepsilon}{\sqrt{1+t}} \times \frac{1}{\langle t-r\rangle^{1-\mu}}, \quad (t, r, \omega) \in [0, \infty) \times [0, \infty) \times \mathbb{S}^1.$$

We set $r = |x|$, $\omega = (\omega_1, \omega_2) = x/|x|$ for $x \in \mathbb{R}^2$,

$$S := t\partial_t + x_1\partial_1 + x_2\partial_2,\ L_j := t\partial_j + x_j\partial_t,\ \Omega := x_1\partial_2 - x_2\partial_1.$$

Then we have

$$[\Box, S] = 2\Box, \quad [\Box, L_j] = [\Box, \Omega] = 0, \quad j = 1, 2.$$

We also set $\partial_r = \omega_1\partial_1 + \omega_2\partial_2$ and $\partial^{\pm} = \partial_t \pm \partial_r$, we get

$$\partial^+ = \frac{1}{t+r}\left(S + \omega_1 L_1 + \omega_2 L_2\right).$$

For $r \sim t \gg 1$, we get followings:

- $\partial_t = \frac{1}{2}(\partial^+ + \partial^-) \sim -\frac{-1}{2}\partial^-, \quad \partial_r = \frac{1}{2}(\partial^+ - \partial^-) \sim -\frac{1}{2}\partial^-.$
- $\partial_1 = \omega_1 \partial_r - \frac{\omega_2}{r}\Omega \sim -\frac{\omega_1}{2}\partial^-.$
- $\partial_2 = \omega_2 \partial_r + \frac{\omega_1}{r}\Omega \sim -\frac{\omega_2}{2}\partial^-.$

$$\mathcal{D} = -\frac{1}{2}\partial^-, \ \omega_0 = -1 \Longrightarrow \ \partial_a \sim \omega_a \mathcal{D}, \quad a = 0, 1, 2.$$

Lemma 4

$$\left| |x|^{1/2}\partial\phi(t,x) - \hat{\omega}(x)\mathcal{D}\left(|x|^{1/2}\phi(t,x)\right) \right| \le C\langle t + |x| \rangle^{-1/2} |\Gamma\phi(t,x)|$$

for $(t,x) \in \Lambda_\infty = \{(t,x) \in [0,\infty) \times \mathbb{R}^2; |x| \ge t/2 \ge 1\}$,
where $\Gamma = (S, L_1, L_2, \Omega, \partial_0, \partial_1, \partial_2)$.

We also have the relation

$$\partial^+\partial^-(r^{1/2}\phi) = r^{1/2}\Box\phi + \frac{1}{4r^{3/2}}(4\Omega^2+1)\phi.$$

Now we set $U(t,x) = \mathcal{D}(r^{1/2}u(t,x))$, then we obtain

$$\partial^+U(t,x) = -\frac{P(\omega)}{2t}U(t,x)^3 + H(t,x),$$

where $H = \frac{1}{2}\left(r^{1/2}F(\partial u) - \frac{1}{t}P(\omega)U^3\right) - \frac{1}{8r^{3/2}}(4\Omega^2+1)u.$

Remark

- Under (**QN**), H can be regarded as a remainder

$$|H(t,x)| \le C\varepsilon\langle t-r\rangle^{-\mu-1/2}t^{2\mu-3/2}.$$

for $(t,x) \in \Lambda_{\infty,R} = \{(t,x)\in\Lambda_\infty; |x| \le t+R\}$,

Moreover, for $t \geq 2$, $r \geq 0$, $\omega \in \mathbb{S}^1$, we put
$V(t;\sigma,\omega) = U(t, r\omega)|_{r=t+\sigma}$ and $G(t;\sigma,\omega) = H(t, r\omega)|_{r=t+\sigma}$, then

$$(\mathrm{NLW}) \Longrightarrow \partial_t V(t) = -\frac{P(\omega)}{2t} V(t)^3 + G(t).$$

Let $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^1$ be fixed, and we set $\Phi(t;\sigma,\omega) = P(\omega)V(t;\sigma,\omega)^2$.
Then we obtain

$$\begin{aligned} \partial_t \Phi(t) &= 2P(\omega)V(t)\partial_t V(t) \\ &= -\frac{P(\omega)^2}{t} V(t)^4 + 2P(\omega)V(t)G(t) \\ &\leq -\frac{1}{t}\Phi(t)^2 + \frac{C\varepsilon^2}{t^{3/2-2\mu}\langle\sigma\rangle^{3/2}}, \end{aligned}$$

whence we have

$$0 \leq \Phi(t;\sigma,\omega) \leq \frac{C}{\log t}.$$

Therefore we deduce that

$$|V(t;\sigma,\omega)| \le \sqrt{\frac{\Phi(t;\sigma,\omega)}{P(\omega)}} \le \frac{C}{\sqrt{P(\omega)\log t}}.$$

$$\begin{aligned} r^{1/2}|\partial u(t,r\omega)| &\le \sqrt{2}|V(t;r-t,\omega)| + \left|r^{1/2}\partial u(t,r\omega) - \hat{\omega}U(t,r\omega)\right| \\ &\le \frac{C}{\sqrt{P(\omega)\log t}} + \frac{C\varepsilon}{\langle t+r\rangle^{1-\mu}} \end{aligned}$$

for $(t,r\omega)\in\Lambda_{\infty,R}$, whence

$$\begin{aligned} |\partial u(t,r\omega)| &\le \frac{C}{\sqrt{rP(\omega)\log t}}\left(1 + \frac{\varepsilon\sqrt{P(\omega)\log t}}{t^{1-\mu}}\right) \\ &\le \frac{C\varepsilon}{\sqrt{t}}\cdot\frac{1}{\sqrt{P(\omega)\varepsilon^2\log t}} \end{aligned}$$

for $(t,r\omega)\in\Lambda_{\infty,R}$. □

Lemma 1 (Key estimate)

For $t \geq 2$, $r \geq 0$, $\omega \in \mathbb{S}^1$, we have

$$|\partial u(t, r\omega)| \leq \frac{C\varepsilon}{\sqrt{t}} \min \left\{ \frac{1}{\sqrt{P(\omega)\varepsilon^2 \log t}}, \frac{1}{\langle t-r \rangle^{1-\mu}} \right\}.$$

By using Lemma 1, we obtain

$$\begin{aligned}|\partial u(t, r\omega)| &\leq \frac{C\varepsilon}{\sqrt{t}} \left(\frac{1}{\sqrt{P(\omega)\varepsilon^2 \log t}} \right)^{2\lambda} \left(\frac{1}{\langle t-r \rangle^{1-\mu}} \right)^{1-2\lambda} \\ &= \frac{C\varepsilon}{(\varepsilon^2 \log t)^{\lambda}} \cdot \frac{1}{P(\omega)^{\lambda}} \cdot \frac{1}{\sqrt{t}\langle t-r \rangle^{(1-\mu)(1-2\lambda)}}\end{aligned}$$

for $\lambda > 0$, $t \geq 2$, $r \geq 0$ and $\omega \in \mathbb{S}^1$,

Proof of main theorem

- We can take $R > 0$ such that $\operatorname{supp} f \cup \operatorname{supp} g \subset \{x \in \mathbb{R}^2; |x| \le R\}$. Then, from the finite propagation property, we get

$$\operatorname{supp} u(t, \cdot) \subset \{x \in \mathbb{R}^2; |x| \le t + R\}.$$

- Let $\rho = (\varepsilon^2 \log t)^{\frac{2\lambda}{1-2\mu}}$. Then we can split

$$\begin{aligned} 2\|u(t)\|_E^2 &= \int_{|x|\le t+R} |\partial u(t,x)|^2 dx \\ &= \int_{|x|\le t+R-\rho} |\partial u(t,x)|^2 dx + \int_{t+R-\rho\le|x|\le t+R} |\partial u(t,x)|^2 dx \\ &=: I_1 + I_2. \end{aligned}$$

Estimate for I_1

- By Lemma 1, we have

$$|\partial u(t, r\omega)| \le \frac{C\varepsilon}{\sqrt{t}\langle t-r\rangle^{1-\mu}}, \quad t \ge 2,\ r \ge 0,\ \omega \in \mathbb{S}^1.$$

Then, for $t \ge 2$, we get

$$\begin{aligned} I_1 &= \int_{|x|\le t+R-\rho} |\partial u(t,x)|^2 dx \\ &\le \int_0^{2\pi} \int_0^{t+R-\rho} \frac{C\varepsilon^2}{t\langle t-r\rangle^{2-2\mu}}\, r dr d\theta \\ &\le C\varepsilon^2 \int_0^{t+R-\rho} \frac{dr}{(R+t-r)^{2-2\mu}} \\ &\le \frac{C\varepsilon^2}{\rho^{1-2\mu}}. \end{aligned}$$

Estimate for I_2

Since we have

$$|\partial u(t, r\omega)| \le \frac{C\varepsilon}{(\varepsilon^2 \log t)^\lambda} \cdot \frac{1}{P(\omega)^\lambda} \cdot \frac{1}{\sqrt{t}\langle t-r\rangle^{(1-\mu)(1-2\lambda)}},$$

for $t \ge 2$, we obtain

$$I_2 = \int_{t+R-\rho\le|x|\le t+R} |\partial u(t,x)|^2 dx$$

$$\le \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}} \left(\int_0^{2\pi} \frac{d\theta}{P(\cos\theta, \sin\theta)^{2\lambda}} \right) \left(\int_{t+R-\rho}^{t+R} \frac{r dr}{t\langle t-r\rangle^{2(1-\mu)(1-2\lambda)}} \right)$$

$$\le \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}} \left(\int_0^{2\pi} \frac{d\theta}{P(\cos\theta, \sin\theta)^{2\lambda}} \right) \left(\int_{\mathbb{R}} \frac{d\sigma}{\langle\sigma\rangle^{2(1-\mu)(1-2\lambda)}} \right).$$

Lemma 2

Suppose that $\Psi(\theta)$ is a real-valued function on $[0, 2\pi]$ which can be written as a (finite) linear combination of the terms $\cos^{p_1}\theta \sin^{p_2}\theta$ with p_1, $p_2 \in \mathbb{Z}_{\geq 0}$. If $\Psi(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$, then we have either of the following three assertions:

(a) $\Psi(\theta) = 0$ for all $\theta \in [0, 2\pi]$.

(b) $\Psi(\theta) > 0$ for all $\theta \in [0, 2\pi]$.

(c) There exist positive integers m, $\nu_1, \ldots, \nu_m$, points $\theta_1, \ldots, \theta_m \in [0, 2\pi]$, and positive constants $c_1, \ldots, c_m$ such that
- $\Psi(\theta) > 0$ for $\theta \in [0, 2\pi] \setminus \{\theta_1, \ldots, \theta_m\}$,
- $\Psi(\theta) = (\theta - \theta_j)^{2\nu_j}\big(c_j + o(1)\big)$ as $\theta \to \theta_j$ for each $j = 1, \ldots, m$.

Let $\Psi(\theta) = P(\cos\theta, \sin\theta)$. $\mathbb{S}^1 \ni \omega = (\cos\theta, \sin\theta)$

- $P(\cos\theta, \sin\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ $\iff$ **(A)**

(a) $P(\cos\theta, \sin\theta) = 0$ for all $\theta \in [0, 2\pi]$ $\iff$ **(CN)**

(b) $P(\cos\theta, \sin\theta) > 0$ for all $\theta \in [0, 2\pi]$ $\iff$ **(A$_+$)**

(c)
- $P(\cos\theta, \sin\theta) > 0$ for $\theta \in [0, 2\pi] \setminus \{\theta_1, \ldots, \theta_m\}$
- $P(\cos\theta, \sin\theta) = (\theta - \theta_j)^{2\nu_j}\big(c_j + o(1)\big)$ as $\theta \to \theta_j$

$\iff$ **(A)** ○, **(CN)** ×, **(A$_+$)** ×.

We focus on the case $(\mathbf{c})$

$(\mathbf{c})$
- $P(\cos\theta, \sin\theta) > 0$ for $\theta \in [0, 2\pi] \backslash \{\theta_1, \dots, \theta_m\}$
- $P(\cos\theta, \sin\theta) = (\theta - \theta_j)^{2\nu_j}\big(c_j + o(1)\big)$ as $\theta \to \theta_j$

Lemma 3

Assume that $P(\cos\theta, \sin\theta)$ satisfies $(\mathbf{c})$. We set $\nu = \max\{\nu_1, \dots, \nu_m\}$. Then, for $0 < \gamma < 1/(2\nu)$, we have

$$\int_0^{2\pi} \frac{d\theta}{P(\cos\theta, \sin\theta)^\gamma} < \infty.$$

$\because$ We can take $\delta > 0$ so small that

$$P(\cos\theta, \sin\theta) \geq \frac{c_j}{2}(\theta - \theta_j)^{2\nu_j} \text{ for } \theta \in (\theta_j - \delta, \theta_j + \delta).$$

□

Estimate for I_2

- We apply Lemma 3 with $\gamma = 2\lambda$,

$$0 < \lambda < 1/(4\nu) \implies \int_0^{2\pi} \frac{d\theta}{P(\cos\theta, \sin\theta)^{2\lambda}} < \infty.$$

- With this λ, we choose

$$0 < \mu < \min\left\{\frac{1}{10}, \frac{1-4\lambda}{2-4\lambda}\right\} \implies 2(1-\mu)(1-2\lambda) > 1.$$

Then, for $t \geq 2$, we obtain

$$I_2 \leq \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}} \left(\int_0^{2\pi} \frac{d\theta}{P(\cos\theta, \sin\theta)^{2\lambda}}\right)\left(\int_{\mathbb{R}} \frac{d\sigma}{\langle\sigma\rangle^{2(1-\mu)(1-2\lambda)}}\right)$$

$$\leq \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}}.$$

Estimate for the energy

Eventually, for $t \geq 2$, we have, $\rho = (\varepsilon^2 \log t)^{\frac{2\lambda}{1-2\mu}}$

$$\|u(t)\|_E^2 \leq \frac{C\varepsilon^2}{\rho^{1-2\mu}} + \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}} \leq \frac{C\varepsilon^2}{(\varepsilon^2 \log(t+2))^{2\lambda}}.$$

On the other hands, for $t \geq 0$, we get

$$\|u(t)\|_E^2 \leq C\varepsilon^2 \int_0^{t+R} \frac{rdr}{t\langle t-r\rangle^{2-2\mu}} \leq C\varepsilon^2 \int_{\mathbb{R}} \frac{d\sigma}{\langle \sigma \rangle^{2-2\mu}} \leq C\varepsilon^2.$$

Summing up them, we obtain

$$\|u(t)\|_E \leq \frac{C\varepsilon}{(1+\varepsilon^2 \log(t+2))^{\lambda}}$$

for $t \geq 0$. □

Remarks on the decay rate

We can take $\lambda = 1/(4\nu) - \delta$ with arbitrarily small $\delta > 0$, and 2ν is the maximum of the vanishing order of zeros of $P(\cos\theta, \sin\theta)$.

$F_c(\partial u)$	$P(\cos\theta, \sin\theta)$		λ
$-(\partial_1 u)^2 \partial_t u$	$\cos^2\theta$	$(\theta - \pi/2)^2 + \cdots$	$1/4 - \delta$
		$(\theta - 3\pi/2)^2 + \cdots$	
$-(\partial_1 u)^2(\partial_t u + \partial_2 u)$	$\cos^2\theta(1 - \sin\theta)$	$(1/2)(\theta - \pi/2)^4 + \cdots$	$1/8 - \delta$
		$2(\theta - 3\pi/2)^2 + \cdots$	
$-(\partial_t u + \partial_2 u)^3$	$(1 - \sin\theta)^3$	$(1/8)(\theta - \pi/2)^6 + \cdots$	$1/12 - \delta$