Positive solutions of the $\mathcal{A}$-Laplace equation with a potential

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Himeji conference on PDEs
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March 2 – 4, 2022

Joint work with Yongjun Hou and Antti Rasila (GTIIT)

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It extends works of YP-Tintarev (2007), YP-Regev (2015), and YP-Psaradakis (2016).

Our aim

We would like to discuss some local and global properties of positive solutions of the quasilinear homogeneous equation

$$Q'_{p,\mathcal{A},V}[u] \triangleq -\operatorname{div}\mathcal{A}(x,\nabla u) + V(x)|u|^{p-2}u = 0 \qquad \text{in } \Omega,$$

where $\Omega \subseteq \mathbb{R}^n$ is a nonempty domain, and $1 < p < \infty$, V is a given potential, and the principal part

$$\Delta_{\mathcal{A}}[u] \triangleq \operatorname{div}\mathcal{A}(x,\nabla u)$$

is the $\mathcal{A}$-Laplacian studied in the influential book "Nonlinear potential theory of degenerate elliptic equations" by J. Heinonen, T. Kilpeläinen, and O. Martio.

The $\mathcal{A}$-Laplacian

Assumptions

The function $\mathcal{A}(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following properties:

- **Regularity:** *For a.e. $x \in \Omega$, the function $\mathcal{A}(x,\xi) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to ξ, and the function $x \mapsto \mathcal{A}(x,\xi)$ is Lebesgue measurable in Ω for all $\xi \in \mathbb{R}^n$.*
- $(p-1)$**-Homogeneity in ξ:** *For all $\lambda \in \mathbb{R} \setminus \{0\}$,*

$$\mathcal{A}(x, \lambda\xi) = \lambda\,|\lambda|^{p-2}\,\mathcal{A}(x,\xi).$$

- **Ellipticity:** *For all $\omega \Subset \Omega$, all $\xi \in \mathbb{R}^n$, and a.e. $x \in \omega$,*

$$\alpha_\omega |\xi|^p \le \mathcal{A}(x,\xi)\cdot\xi \le \beta_\omega\,|\xi|^p.$$

- **Monotonicity:** *For a.e. $x \in \Omega$ and all $\xi \ne \eta \in \mathbb{R}^n$,*

$$\big(\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta)\big)\cdot(\xi-\eta) > 0.$$

The $\mathcal{A}$-Laplacian

Note that $|\xi|_{\mathcal{A}}^p \triangleq \mathcal{A}(x,\xi)\cdot\xi$ is a strictly convex function with respect to ξ. i.e.,

$$|\xi|_{\mathcal{A}}^p - |\eta|_{\mathcal{A}}^p - p\mathcal{A}(x,\eta)\cdot(\xi-\eta) > 0 \qquad \forall \xi \neq \eta \in \mathbb{R}^n.$$

Lemma

For a.e. $x \in \Omega$, the function $|\xi|_{\mathcal{A}} = (\mathcal{A}(x,\xi)\cdot\xi)^{1/p}$ is a norm on $\mathbb{R}^n$.

The $\mathcal{A}$-Laplacian

Note that $|\xi|_{\mathcal{A}}^{p} \triangleq \mathcal{A}(x,\xi)\cdot\xi$ is a strictly convex function with respect to ξ. i.e.,

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Lemma

For a.e. $x \in \Omega$, the function $|\xi|_{\mathcal{A}} = (\mathcal{A}(x,\xi)\cdot\xi)^{1/p}$ is a norm on $\mathbb{R}^n$.

Lemma (Generalized Hölder inequality)

Let p' be the conjugate exponent of p. For every $\xi, \eta \in \mathbb{R}^n$

$$\left|\mathcal{A}(x,\xi)\cdot\eta\right| \leq \left(\mathcal{A}(x,\xi)\cdot\xi\right)^{1/p'}\left(\mathcal{A}(x,\eta)\cdot\eta\right)^{1/p} = |\xi|_{\mathcal{A}}^{p-1}|\eta|_{\mathcal{A}}.$$

Local strong convexity of $|\xi|_{\mathcal{A}}^{p}$

For some results we further need to assume:

Assumption (Local strong convexity of $|\xi|_{\mathcal{A}}^{p}$)

We suppose that $|\xi|_{\mathcal{A}}^{p}$ is a locally strongly convex function with respect to ξ, that is, there exists $\bar{p} \geq p$ such that for every domain $\omega \Subset \Omega$ there exists a positive constant $C_\omega(\bar{p}, \mathcal{A})$ such that

$$|\xi|_{\mathcal{A}}^{p} - |\eta|_{\mathcal{A}}^{p} - p\mathcal{A}(x,\eta)\cdot(\xi-\eta) \geq C_\omega(\bar{p},\mathcal{A})|\xi-\eta|_{\mathcal{A}}^{\bar{p}} \qquad \forall \xi,\eta \in \mathbb{R}^n.$$

(p, A)-Laplacian

Example

- For a symmetric, locally bounded, and locally uniformly positive definite matrix function $A(x)$ defined on Ω, the operator

$$\Delta_{p,A}[u] = \operatorname{div}(|\nabla u|_A^{p-2} A\nabla u)$$

is called the (p, A)-Laplacian, where $|\xi|_A = \sqrt{A\xi \cdot \xi}$.

- If A is the identity matrix, then

$$\Delta_p[u] = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is the celebrated p-Laplacian.

(p, A)-Laplacian - local strong convexity

Lemma (Pinchover-Tintarev (2007))

For the (p, A)-Laplacian and in particular, for the p-Laplacian, there exists a constant $C(p) > 0$ such that for all $\xi, \eta \in \mathbb{R}^n$ ($\eta \neq 0$ if $p < 2$) and a.e. $x \in \Omega$,

$$|\xi|_A^p - |\eta|_A^p - p|\eta|_A^{p-2}\eta \cdot (\xi - \eta) \geq C(p)[\xi, \eta]_{p,A},$$

where

$$[\xi, \eta]_{p,A} \triangleq \begin{cases} |\xi - \eta|_A^p & \text{if } p \geq 2, \\ |\xi - \eta|_A^2(|\xi|_A + |\eta|_A)^{p-2} & \text{if } 1 < p < 2. \end{cases}$$

Pseudo-p-Laplacian

Example

Let $p \geq 2$ and $a_i \asymp 1$ locally in Ω, for all $i = 1, 2, \ldots, n$,
Consider $\mathcal{A}(x, \xi) = (a_1(x)|\xi_1|^{p-2}\xi_1, \ldots, a_n(x)|\xi_n|^{p-2}\xi_n)$. For a.e. $x \in \Omega$ and every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, we have

(1) $|\xi|_{\mathcal{A}}^p = \sum_{i=1}^n a_i(x)|\xi_i|^p$,

(2) the operator $\mathcal{A}$ satisfies the basic assumptions above, and the local strong convexity.

Definition (Pseudo-p-Laplacian)

If $a_i = 1$ for all $i = 1, 2, \ldots, n$, then the corresponding operator

$$\tilde{\Delta}_p[u] = \sum_{i=1}^{n} \partial_i(|\partial_i u|^{p-2}\partial_i u)$$

is called the pseudo-p-Laplacian studied by Francoise Demengel and others.

Morrey potentials

Definition

For every domain $\omega \Subset \Omega$, $M^q(p;\omega)$ is a Morrey space with $q = q(p,n) \in [1,\infty]$. Furthermore, $M^q_{\mathrm{loc}}(p;\Omega) \triangleq \cap_{\omega \Subset \Omega} M^q(p;\omega)$, which is called the local Morrey space.

Remark

1. $L^q_{\mathrm{loc}}(\Omega) \subseteq M^q_{\mathrm{loc}}(p;\Omega) \subseteq L^1_{\mathrm{loc}}(\Omega)$.

2. We assume that the potential V satisfies $V \in M^q_{\mathrm{loc}}(p;\Omega)$.

Morrey-Adams theorem

The following key estimates assert that $\int_\omega |V||u|^p \,\mathrm{d}x$ has zero relative bound with respect to the p-Laplacian.

Theorem (Morrey-Adams estimate)

Let $\omega \Subset \mathbb{R}^n$ be a domain and $V \in M^q(p;\omega)$.

(1) *There exists a constant $C(n,p,q) > 0$ such that for any $\delta > 0$ and all $u \in W_0^{1,p}(\omega)$,*

$$\int_\omega |V||u|^p \,\mathrm{d}x \le \delta \|\nabla u\|^p_{L^p(\omega;\mathbb{R}^n)} + \frac{C(n,p,q)}{\delta^{n/(pq-n)}} \|V\|^{pq/(pq-n)}_{M^q(p;\omega)} \|u\|^p_{L^p(\omega)}.$$

(2) *For any $\omega' \Subset \omega$ with Lipschitz boundary, there exists δ_0 such that for any $0 < \delta \le \delta_0$ and all $u \in W^{1,p}(\omega')$, with $C = C\left(n,p,q,\omega',\omega,\delta,\|V\|_{M^q(p;\omega)}\right)$,*

$$\int_{\omega'} |V||u|^p \,\mathrm{d}x \le \delta \|\nabla u\|^p_{L^p(\omega';\mathbb{R}^n)} + C\|u\|^p_{L^p(\omega')}.$$

The variational functional $Q_{p,\mathcal{A},V}[\varphi;\omega]$

Notation

For a domain $\omega \subseteq \Omega$ *and* $\varphi \in W_c^{1,p}(\omega)$ *, let*

$$Q^{\omega}_{p,\mathcal{A},V}[\varphi] \triangleq \int_\omega \left(|\nabla\varphi|^p_{\mathcal{A}} + V|\varphi|^p\right) \mathrm{d}x.$$

Definition

The functional $Q^{\Omega}_{p,\mathcal{A},V}$ *is said to be nonnegative in* Ω *if* $Q^{\Omega}_{p,\mathcal{A},V}[\varphi] \geq 0$ *for all* $\varphi \in W_c^{1,p}(\Omega)$*.*

Definition ($\mathcal{A}$-Laplace equation with Morrey potential)

Let $g \in L^1_{\mathrm{loc}}(\Omega)$. $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a (weak) solution of the equation

$$Q'_{p,\mathcal{A},V}[v] \triangleq -\operatorname{div} \mathcal{A}(x, \nabla v) + V|v|^{p-2}v = g,$$

in Ω if for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \mathcal{A}(x, \nabla v) \cdot \nabla\varphi \, \mathrm{d}x + \int_\Omega V|v|^{p-2}v\varphi \, \mathrm{d}x = \int_\Omega g\varphi \, \mathrm{d}x,$$

and a supersolution of $Q'_{p,\mathcal{A},V}[u] = g$ in Ω, if for all $0 \leq \varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \mathcal{A}(x, \nabla v) \cdot \nabla\varphi \, \mathrm{d}x + \int_\Omega V|v|^{p-2}v\varphi \, \mathrm{d}x \geq \int_\Omega g\varphi \, \mathrm{d}x.$$

A supersolution $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the equation is said to be proper if v is not a solution.

Harnack inequality & Hölder continuity

Theorem

Let $V \in M^q_{\mathrm{loc}}(p;\Omega)$. Let v be a nonnegative solution v of $Q'_{p,\mathcal{A},V}[u] = 0$ in a domain $\omega \Subset \Omega$. Then for any $\omega' \Subset \omega$,

$$\sup_{\omega'} v \leq C \inf_{\omega'} v,$$

where C is a positive constant depending only on $n, p, q, \omega', \omega, \alpha_\omega, \beta_\omega$, and $\|V\|_{M^q(p;\omega)}$ (but not on v).
Moreover, solutions of such equation satisfies local Hölder estimates.
In particular, the Harnack convergence principle holds true.

Generalized principal eigenvalue

Definition

The generalized principal eigenvalue of $Q'_{p,\mathcal{A},V}$ in a domain $\omega \subseteq \Omega$ is defind by

$$\lambda_1 = \lambda_1(Q^{\omega}_{p,\mathcal{A},V}) \triangleq \inf_{u \in W^{1,p}_c(\omega)\setminus\{0\}} \frac{Q^{\omega}_{p,\mathcal{A},V}[u]}{\|u\|^p_{L^p(\omega)}} .$$

Generalized principal eigenvalue

Definition

The generalized principal eigenvalue of $Q'_{p,\mathcal{A},V}$ in a domain $\omega \subseteq \Omega$ is defind by

$$\lambda_1 = \lambda_1(Q^{\omega}_{p,\mathcal{A},V}) \triangleq \inf_{u \in W^{1,p}_c(\omega)\setminus\{0\}} \frac{Q^{\omega}_{p,\mathcal{A},V}[u]}{\|u\|^p_{L^p(\omega)}} .$$

Remark

It follows that for a domain $\omega \Subset \Omega$ we have

$$\lambda_1(Q^{\omega}_{p,\mathcal{A},V}) = \inf_{u \in W^{1,p}_0(\omega)\setminus\{0\}} \frac{Q^{\omega}_{p,\mathcal{A},V}[u]}{\|u\|^p_{L^p(\omega)}} .$$

Principal eigenvalue of $Q'_{p,\mathcal{A},V}$

Definition

Let $\omega \Subset \Omega$ be a domain and $V \in M^q(p;\omega)$. A real number λ is called an eigenvalue with an eigenfunction v of the Dirichlet eigenvalue problem

$$\begin{cases} Q'_{p,\mathcal{A},V}[w] = \lambda |w|^{p-2}w & \text{in } \omega, \\ w = 0 & \text{on } \partial\omega, \end{cases}$$

if $v \in W_0^{1,p}(\omega) \setminus \{0\}$ satisfies the above equation in the weak sense.

A principal eigenfunction is a nonnegative eigenfunction associated to a principal eigenvalue.

Principal eigenvalue of $Q'_{p,\mathcal{A},V}$ in bounded domains

Theorem

Let $\omega \Subset \Omega$ be a bounded subdomain. Then

(1) *the generalized principal eigenvalue $\lambda_1(Q^{\omega}_{p,\mathcal{A},V})$ is the unique principal eigenvalue of the operator $Q'_{p,\mathcal{A},V}$,*

(2) *the principal eigenvalue is simple.*

Generalized weak/strong maximum principle

Definition

Let $\omega \Subset \Omega$ be a Lipschitz domain.

- *The operator $Q'_{p,\mathcal{A},V}$ is said to satisfy the generalized weak maximum principle in ω if every supersolution v of the equation $Q'_{p,\mathcal{A},V}[u] = g$ in ω with $0 \leq g \in L^{p'}(\omega)$ and $v \geq 0$ on $\partial\omega$ is nonnegative in ω;*
- *the operator $Q'_{p,\mathcal{A},V}$ satisfies the generalized strong maximum principle in ω if any such a supersolution v is either the zero function or strictly positive in ω.*

Theorem

Let $\omega \Subset \Omega$ be a Lipschitz subdomain. Then TFAAE:

(1) *The operator $Q'_{p,\mathcal{A},V}$ satisfies the generalized weak/strong maximum principle in ω.*

(2) *The principal eigenvalue $\lambda_1 = \lambda_1(Q_{p,\mathcal{A},V};\omega)$ is positive.*

(3) *The equation $Q'_{p,\mathcal{A},V}[u] = 0$ has a proper positive supersolution in $W_0^{1,p}(\omega)$.*

(3′) *The equation $Q'_{p,\mathcal{A},V}[u] = 0$ has a proper positive supersolution in $W^{1,p}(\omega)$.*

(4) *For any nonnegative $g \in L^{p'}(\omega)$, there exists a nonnegative solution, which is either zero or positive, of the equation $Q'_{p,\mathcal{A},V}[u] = g$ in $W_0^{1,p}(\omega)$.*

Moreover, if the local strong convexity is satisfied, then the solution in (4) is unique.

Agmon-Allegretto-Piepenbrink (AAP) type theorem

Theorem (AAP type theorem)

The following three assertions are equivalent:

(1) *the functional* $Q^{\Omega}_{p,\mathcal{A},V}$ *is nonnegative in* Ω*;*

(2) *the equation* $Q'_{p,\mathcal{A},V}[w] = 0$ *admits a positive solution* $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$*;*

(3) *the equation* $Q'_{p,\mathcal{A},V}[w] = 0$ *admits a positive supersolution* $\tilde{v} \in W^{1,p}_{\mathrm{loc}}(\Omega)$*.*

Criticality/subcriticality

Definition

- If there exists a nonnegative function $W \in M^q_{\mathrm{loc}}(p;\Omega) \setminus \{0\}$ such that

$$Q^{\Omega}_{p,\mathcal{A},V}[\varphi] \geq \int_{\Omega} W|\varphi|^p \,\mathrm{d}x,$$

  for all $\varphi \in C_c^{\infty}(\Omega)$, we say that the functional $Q^{\Omega}_{p,\mathcal{A},V}$ is subcritical in Ω, and W is a Hardy-weight for $Q^{\Omega}_{p,\mathcal{A},V}$ in Ω;
- if $Q^{\Omega}_{p,\mathcal{A},V}$ is nonnegative in Ω but $Q^{\Omega}_{p,\mathcal{A},V}$ does not admit a Hardy-weight in Ω, we say that the functional $Q^{\Omega}_{p,\mathcal{A},V}$ is critical in Ω;
- if $Q^{\Omega}_{p,\mathcal{A},V}$ is not nonnegative in Ω, we say that the functional $Q^{\Omega}_{p,\mathcal{A},V}$ is supercritical in Ω.

Null-sequences and ground states

Definition

A nonnegative sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ *is called a* *null-sequence* *with respect to the nonnegative functional* $Q^\Omega_{p,\mathcal{A},V}$ *if*

- *there exists a fixed open set* $U \Subset \Omega$ *such that* $\|u_k\|_{L^p(U)} = 1$ *for all* $k \in \mathbb{N}$*;*
- *and* $\lim_{k\to\infty} Q^\Omega_{p,\mathcal{A},V}[u_k] = 0$.

Definition

A *ground state* *of the nonnegative functional* $Q^\Omega_{p,\mathcal{A},V}$ *is a positive function* $\phi \in W^{1,p}_{\mathrm{loc}}(\Omega)$*, which is an* $L^p_{\mathrm{loc}}(\Omega)$ *limit of a null-sequence.*

$(\mathcal{A}, V)$-capacity

Definition

Assume that the functional $Q^{\Omega}_{p,\mathcal{A},V}$ is nonnegative in Ω). For every compact subset K of Ω, we define the $(\mathcal{A}, V)$-capacity of K in Ω as

$$\mathrm{Cap}_{\mathcal{A},V}(K,\Omega) \triangleq \inf\left\{ Q^{\Omega}_{p,\mathcal{A},V}[\varphi] \mid \varphi \in C^{\infty}_{c}(\Omega), \varphi \geq 1 \text{ on } K \right\}.$$

Positive solution of minimal growth

Theorem

Let $x_0 \in \Omega$. If $Q^{\Omega}_{p,\mathcal{A},V} \geq 0$ in Ω, then there exists a positive solution u_{x_0} of the equation $Q'_{p,\mathcal{A},V}[w] = 0$ in $\Omega \setminus \{x_0\}$ that has minimal growth in a neighborhood of infinity in Ω, i.e. for any smooth compact subset $K \Subset \Omega$ with $\{x_0\} \in \mathring{K}$, any positive supersolution v of $Q'_{p,\mathcal{A},V}[w] = 0$ in $\Omega \setminus K$ such that $u_{x_0} \leq v$ on ∂K, satisfies $u \leq v$ in $\Omega \setminus K$.

Characterizations of criticality

Theorem

Let $\mathcal{A}$ satisfy the local strong convexity and the functional $Q^{\Omega}_{p,\mathcal{A},V}$ be nonnegative in Ω. Then TFAAE:

- *the functional $Q^{\Omega}_{p,\mathcal{A},V}$ is critical in Ω;*
- *the functional $Q^{\Omega}_{p,\mathcal{A},V}$ has a null-sequence in $C_c^{\infty}(\Omega)$ converging locally uniformly to a ground state;*
- *the functional $Q^{\Omega}_{p,\mathcal{A},V}$ has a ground state;*
- *the equation $Q'_{p,\mathcal{A},V}[u] = 0$ has a unique positive supersolution;*
- $\mathrm{Cap}_{\mathcal{A},V}(B,\Omega) = 0$ *of all closed balls $B \Subset \Omega$;*
- *u_{x_0} has a removable singularity. u_{x_0} is called the global minimal positive solution in Ω.*

Removability of isolated singularity

Theorem

Let $p \leq n$ and $x_0 \in \Omega$. Consider a positive solution u of $Q'_{p,\mathcal{A},V}[w] = 0$ in a punctured neighborhood of x_0.

(1) *If u is bounded in some punctured neighborhood of x_0, then u can be extended to a nonnegative solution in Ω.*

(2) *Otherwise, $\lim_{x \to x_0} u(x) = \infty$.*

Minimal positive Green functions

Definition

u_{x_0} is called a *minimal positive Green function of $Q'_{p,\mathcal{A},V}$ in Ω with singularity* *at x_0, if u_{x_0} admits a nonremovable singularity at x_0. We denote such a Green function by $G^{\Omega}_{\mathcal{A},V}(x, x_0)$.*

Minimal positive Green functions

Definition

u_{x_0} is called a minimal positive Green function of $Q'_{p,\mathcal{A},V}$ in Ω with singularity at x_0, if u_{x_0} admits a nonremovable singularity at x_0. We denote such a Green function by $G^{\Omega}_{\mathcal{A},V}(x,x_0)$.

Theorem

Let $\mathcal{A}$ satisfy the local strong convexity, and $Q^{\Omega}_{p,\mathcal{A},V} \geq 0$ in Ω.

(1) *If $p \leq n$, and $Q'_{p,\mathcal{A},V}$ admits a minimal positive Green function in Ω, then $Q^{\Omega}_{p,\mathcal{A},V}$ is subcritical in Ω.*

(2) *If $p > n$, and $Q'_{p,\mathcal{A},V}$ admits a minimal positive Green function in Ω with singularity at some point $x_0 \in \Omega$ s.t.*

$$\lim_{x \to x_0} G^{\Omega}_{\mathcal{A},V}(x,x_0) = c \qquad \text{for some positive constant } c,$$

then $Q^{\Omega}_{p,\mathcal{A},V}$ is subcritical in Ω.

Thank you for your attention!