Continuum limit problem of discrete Schrödinger operators[1]

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[1]Based on joint works with Pavel Exner (Czech Technical University) and Shu Nakamura (Gakushuin University)

Contents

This talk is based on the following papers:

- S. Nakamura, Y. T.: On a continuum limit of discrete Schrödinger operators on square lattice. Journal of Spectral Theory **11** (2021), no. 1, 355–367.
- P. Exner, S. Nakamura, Y. T.: Continuum limit of the lattice quantum graph Hamiltonian. Preprint arXiv:2202.06586.

1. Discrete Schrödinger operators on $h\mathbb{Z}^\nu$

2. Schrödinger operators on quantum graphs

Definition of H and H_h

Schrödinger operators

$$H = H_0 + V = -\Delta + V(x) \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^\nu).$$

Square lattice with width $h > 0$

For $v \in \mathcal{H}_2 = \ell^2(h\mathbb{Z}^\nu)$, $\nu = 1, 2, \cdots$, $h > 0$, $V : \mathbb{R}^\nu \to \mathbb{R}$,

$$H_2 v(z) = (H_{2,0} + V)v(z) = -\Delta_d v(z) + V(z)v(z), \quad z \in h\mathbb{Z}^\nu,$$

where $\Delta_d v(z) := \frac{1}{h^2} \sum_{|w-z|=h} (v(w) - v(z))$.

$\mathcal{H}_2$ is equipped with the norm $\|v\|_2 = h^{\frac{\nu}{2}} \left(\sum_{z \in h\mathbb{Z}^\nu} |v(z)|^2 \right)^{\frac{1}{2}}$.

Known results, motivation

Remark Since

$$u(z \pm he_j) = u(z) \pm h\partial_{x_j} u(z) + \tfrac{h^2}{2}\partial_{x_j}^2 u(z) \pm \tfrac{h^3}{6}\partial_{x_j}^3 u(z) + O(h^4)$$

by Taylor's theorem, we formally have $\Delta_d u(z) = \Delta u(z) + O(h^2)$.

Previous results Exner-Hejčík-Šeba (2006), Rabinovich (2013), Hong-Yang (2019), Kirkpatrick-Lenzmann-Staffilani (2013), Bögli-Siegl-Tretter (2017), Strauss (2014).

Related works Cornean-Garde-Jensen (2021), Isozaki-Jensen (2022).

Motivation Asymptotics of H_2 from the view point of spectral theory, e.g. eigenvalues, eigenfunctions, and their converging speed.

Our plan

We choose a suitable $P : \mathcal{H} \to \mathcal{H}_2$ such that $P^*(H_2 - \mu)^{-1}P$ converges to $(H - \mu)^{-1}$ as $h \to 0$ in the operator norm topology (generalized norm resolvent convergence in the textbook of Weidmann (2000)).

$$\begin{array}{ccc} \mathcal{H}_h & \xrightarrow{H_2} & \mathcal{H}_2 \\ P \Big\uparrow & & \Big\downarrow P^* \\ \mathcal{H} & \xrightarrow[H]{} & \mathcal{H} \end{array}$$

Furthermore, we expect P to induce a partial isometry, that is,

$$P^* : \mathcal{H}_2 \hookrightarrow \mathcal{H}, \quad \forall h > 0.$$

Our formulation

Let $\varphi \in \mathcal{S}(\mathbb{R}^\nu)$, and then $P = P_\varphi : \mathcal{H} \to \mathcal{H}_2$ is defined by

$$Pu(z) := h^{-d} \int_{\mathbb{R}^\nu} \overline{\varphi(h^{-1}(x-z))} u(x)dx, \quad h > 0, \ z \in h\mathbb{Z}^\nu.$$

Its adjoint is

$$P^* v(x) = \sum_{z \in h\mathbb{Z}^\nu} \varphi(h^{-1}(x-z))v(z), \quad h > 0, \ v \in \mathcal{H}_2.$$

Lemma 1

P^* is isometric. $\Leftrightarrow$ $\sum_{n \in \mathbb{Z}^\nu} |\hat{\varphi}(\xi + n)|^2 = 1$ for $\xi \in \mathbb{R}^\nu$, where

$$\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) := \int_{\mathbb{R}^\nu} e^{-2\pi i x \cdot \xi} \varphi(x)dx.$$

The proof is given by the Plancherel identity and $\mathcal{F}[\varphi(\cdot + n)](\xi) = e^{2\pi i n \cdot \xi}\hat{\varphi}(\xi)$.

Main result

Theorem 1 (Nakamura-T., Journal of Spectral Theory, 2021)

Let $\varphi \in \mathcal{S}(\mathbb{R}^\nu)$ be fixed so that $\sum_{n\in\mathbb{Z}^\nu} |\hat{\varphi}(\xi+n)|^2 = 1$ and $|\hat{\varphi}(0)| = 1$.
Assume that
$(1)_V$ V is bounded from below.
$(2)_V$ $(V(x)+M)^{-1}$ is uniformly continuous ($M := -\inf_{x\in\mathbb{R}^\nu} V(x) + 1$).
$(3)_V$ V is slowly varying, i.e. $\sup_{|x-y|<1} \frac{V(x)+M}{V(y)+M} < \infty$.
Then, for any $\mu \in \mathbb{C}\backslash\mathbb{R}$,

$$\|P^*(H_2-\mu)^{-1}P - (H-\mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} \to 0,$$
$$\|P(H-\mu)^{-1}P^* - (H_2-\mu)^{-1}\|_{\mathcal{B}(\mathcal{H}_2)} \to 0$$

as $h \to 0$.

Remarks

- If V is smooth, $(3)_V$ is equivalent to $|\nabla V(x)| \leq C(V(x)+M)$.
- Theorem 1 is valid for $V(x)=|x|^a$ and $V(x)=e^{a|x|}$, $a>0$.
- If we set $\hat{\varphi} \in C_c^{\infty}\left((-1,1)^{\nu}\right)$, we obtain

$$V \equiv 0 \implies \|\cdots\| = O(h^2),$$
$$(V(x)-M)^{-1} \in C^{0,\theta}(\mathbb{R}^{\nu}) \implies \|\cdots\| = O(h^{\theta-\varepsilon}), \quad \forall \varepsilon > 0.$$

- We may choose $\varphi = \chi_{[-\frac{1}{2},\frac{1}{2})^{\nu}} \notin \mathcal{S}(\mathbb{R}^{\nu})$, but the converging speed can be slower than the above case.

The argument in the Reed-Simon textbook implies

Corollary 1

Under the assumption of the above theorem,
(1) Let $a, b \in \mathbb{R} \setminus \sigma(H)$, $a < b$. Then $a, b \notin \sigma(H_2)$ for sufficiently small h and

$$\|P^* E_{H_2}((a,b))P - E_H((a,b))\|_{\mathcal{B}(\mathcal{H})} \to 0, \quad h \to 0.$$

(2) Let $d_{\mathsf{H}}(X,Y) = \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\}$ be the Hausdorff distance. Then

$$d_{\mathsf{H}}\left(\sigma((H_2+M)^{-1}), \sigma((H+M)^{-1})\right) \to 0, \quad h \to 0.$$

Each converging speed equals to that of Theorem 1.

Key lemmas

Lemma 2 (Regularity of P)

$\|(1 - P^*P)(H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} \to 0$ as $h \to 0$ for any $\mu \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 3 (Convergence of unperturbed operators and potentials)

(1) $\|(H_{2,0} - \mu)^{-1}P - P(H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_2)} \to 0$ holds.
(2) $(1)_V,\ (2)_V \implies \|(V - \mu)^{-1}P - P(V - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_2)} \to 0.$

Lemma 4 (Relative boundedness)

$(1)_V,\ (3)_V$
$\implies \|H_0(H - \mu)^{-1}\| < \infty,\ \|V(H - \mu)^{-1}\| < \infty$,
$\sup_{h\in(0,1)} \|H_{2,0}(H_2 - \mu)^{-1}\| < \infty,\ \sup_{h\in(0,1)} \|V(H_2 - \mu)^{-1}\| < \infty.$

Proof of Theorem 1

We compute

$$\begin{aligned}
&P^*(H_2-\mu)^{-1}P-(H-\mu)^{-1}\\
&=P^*(H_2-\mu)^{-1}P-P^*P(H-\mu)^{-1}-(1-P^*P)(H-\mu)^{-1}\\
&=P^*(H_2-\mu)^{-1}(PH-H_2P)(H-\mu)^{-1}-(1-P^*P)(H-\mu)^{-1}.
\end{aligned}$$

The second term is estimated by Lemmas 2 and 4:

$$\begin{aligned}
&\|(1-P^*P)(H-\mu)^{-1}\|\\
&\le\|(1-P^*P)(H_0-\mu)^{-1}\|\|(H_0-\mu)(H-\mu)^{-1}\|.
\end{aligned}$$

Proof of Theorem 1

For the first term, we have by triangle inequality and Lemma 4 that

$$\begin{aligned}
&\left\|(H_2-\mu)^{-1}(PH-H_2P)(H-\mu)^{-1}\right\| \\
&\le \left\|(H_2-\mu)^{-1}(PH_0-H_{2,0}P)(H-\mu)^{-1}\right\| \\
&\qquad + \left\|(H_2-\mu)^{-1}(PV-VP)(H-\mu)^{-1}\right\| \\
&\le C\left\|(H_{2,0}-\mu)^{-1}(PH_0-H_{2,0}P)(H_0-\mu)^{-1}\right\| \\
&\qquad + C\left\|(V-\mu)^{-1}(PV-VP)(V-\mu)^{-1}\right\| \\
&= C\left\|(H_{2,0}-\mu)^{-1}P-P(H_0-\mu)^{-1}\right\| \\
&\qquad + C\left\|(V-\mu)^{-1}P-P(V-\mu)^{-1}\right\|,
\end{aligned}$$

which tends to 0 by Lemma 3.

Hilbert space on Quantum graph

We set

$$\mathcal{L} = \{\mathcal{L}_{jn} = [j,n] \mid j,n \in h\mathbb{Z}^\nu, |j-n| = h\},$$

where $[j,n]$ is the line segment connecting j and n. We define

$$\mathcal{H}_1 = L^2(\mathcal{L}) = \bigoplus_{\mathcal{L}_{jn}\in\mathcal{L}} L^2(\mathcal{L}_{jn})$$

with the norm

$$\|\varphi\|_1^2 = \frac{h^{\nu-1}}{\nu} \sum_{\mathcal{L}_{jn}\in\mathcal{L}} \int_{[j,n]} |\varphi_{jn}(t)|^2 dt$$

for $\varphi = (\varphi_{jn}) \in \mathcal{H}_1$. We also set

$$H^1(\mathcal{L}) = \{(\varphi_{jn}) \in \mathcal{H}_1 \mid \varphi_{jn} \in H^1([j,n]), \varphi_{jn}(j) = \varphi_{jm}(j) \text{ for any } j \in h\mathbb{Z}^\nu \text{ and } n,m : \text{neiborhood of } j\}$$

Quantum graph Hamiltonian

Let $V : \mathbb{R}^\nu \to \mathbb{R}$ be bounded from below, and denote $V_j = V(j)$ for $j \in h\mathbb{Z}^\nu$. For $\varphi, \psi \in H^1(\mathcal{L})$, we define

$$q(\varphi, \psi) = \langle \varphi', \psi' \rangle + \sum_{j \in h\mathbb{Z}^\nu} hV_j \varphi_j \overline{\psi_j},$$

where $(\varphi')_{jn}(t) = \frac{d}{dt}\varphi_{jn}(t)$ and $\varphi_j = \varphi_{jn}(j)$.
Let H_1 be the corresponding selfadjoint operator. Then we have

$$\mathcal{D}(H_1) = \{\psi \in \oplus H^2(\mathcal{L}_{jn}) \cap H^1(\mathcal{L}) \mid \sum_{|j-n|=h} \psi'_{jn}(j) = hV_j\psi_j\},$$

$$(H_1\psi)_{jn}(t) = -\psi''_{jn}(t).$$

Main theorem 2

Theorem 2 (Exner-Nakamura-T., arXiv2202.06586)

Under the same assumption of V as Theorem 1, there is a bounded operator $\Psi : \mathcal{H}_1 \to \mathcal{H} = L^2(\mathbb{R}^\nu)$ such that for any $\mu \in \mathbb{C}\backslash\mathbb{R}$,

$$\|(H-\mu)^{-1} - \Psi(\nu H_1 - \mu)^{-1}\Psi^*\|_{\mathcal{B}(\mathcal{H})} \to 0,$$
$$\|(\nu H_1 - \mu)^{-1} - \Psi^*(H-\mu)^{-1}\Psi\|_{\mathcal{B}(\mathcal{H}_1)} \to 0$$

as $h \to 0$.

Theorem 2 implies a similar corollary on approximation of eigenvalues and eigenfunctions, i.e.

$$\|E_H((a,b)) - \Psi E_{H_1}((a,b))\Psi^*\|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{(if } a, b \notin \sigma(H)\text{)},$$
$$d_{\mathsf{H}}\left(\sigma((H-M)^{-1}), \sigma((H_1-M)^{-1})\right) \to 0.$$

Proof of Theorem 2

Let $I : \mathcal{H}_2 \to \mathcal{H}_1$, $\varphi = (\varphi_j) \mapsto I\varphi = (\varphi_{jn})$, be the linear interpolation, i.e.

$$\varphi_{jn}(x(t)) = (1-t)\varphi_j + t\varphi_n, \quad t \in [0,1],$$

where $x(t) = (1-t)j + tn$.
In the following, we prove

Theorem 3

For any $\mu \in \mathbb{C}\backslash\mathbb{R}$,

$$\|(H_2-\mu)^{-1} - I^*(\nu H_1-\mu)^{-1} I\|_{\mathcal{B}(\mathcal{H}_2)} = O(h),$$
$$\|(\nu H_1-\mu)^{-1} - I(H_2-\mu)^{-1} I^*\|_{\mathcal{B}(\mathcal{H}_1)} = O(h).$$

$$\begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{(H_2-\mu)^{-1}} & \mathcal{H}_2 \\ {\scriptstyle I^*}\uparrow & & \downarrow{\scriptstyle I} \\ \mathcal{H}_1 & \xrightarrow[(\nu H_1-\mu)^{-1}]{} & \mathcal{H}_1 \end{array}$$

Proof of Theorem 2

Combining Theorems 1 and 3, we immediately obtain Theorem 2 with $\Psi := P^* I^*$.

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{(\nu H_1-\mu)^{-1}} & \mathcal{H}_1 \\ I\Big\uparrow & & \Big\downarrow I^* \\ \mathcal{H}_2 & \xrightarrow{(H_2-\mu)^{-1}} & \mathcal{H}_2 \\ P\Big\uparrow & & \Big\downarrow P^* \\ \mathcal{H} & \xrightarrow[(H-\mu)^{-1}]{} & \mathcal{H} \end{array}$$

We employ the argument by Exner (1997) and Exner-Hejčík-Šeba (2006) to find the relation between $(H_1-\mu)^{-1}$ and $(H_2-\mu)^{-1}$.
We set the trace operator $K : H^1(\mathcal{L}) \to \mathcal{H}_2$ by

$$K : \varphi = (\varphi_{jn}) \mapsto (K\varphi)_j = \varphi_{jn}(j)$$

Lemma 5

We have

$$K(H_1-\mu)^{-1}I = (H_2-\mu+M_1)^{-1}(1+M_2)^{-1}$$

with some $(H_2-\mu)^{-1}M_j = O(h^2)$, $j=1,\ 2$.

This implies $K(H_1-\mu)^{-1}I = (H_2-\mu)^{-1} + O(h^2)$. Finally Theorem 3 is proved by $\mathcal{D}(H_1) \subset H^1(\mathcal{L})$ and $\|K - I^*\|_{\mathcal{B}(H^1(\mathcal{L}),\mathcal{H}_2)} = O(h)$.

Proof of Lemma 5

It suffices to show that

$$(H_1 - k^2)\psi = I\varphi, \tag{1}$$

$\psi \in \mathcal{D}(H_1)$, $\varphi \in \mathcal{H}_2$, $k^2 \notin \mathbb{R}$ implies $(H_2 - k^2 + M_1)^{-1} K\psi = (1 + M_2)\varphi$. We easily see from (1) that for each jn

$$-\psi''_{jn} - k^2\psi_{jn} = \varphi_{jn} \quad \text{on } \mathcal{L}_{jn}, \tag{2}$$

where $(\varphi_{jn}) = I\varphi$, i.e.

$$\varphi_{jn}(x) = (1 - \frac{x}{h})\varphi_j + \frac{x}{h}\varphi_n = \varphi_j + \frac{x}{h}(\varphi_n - \varphi_j), \quad x \in [0, h] \cong \mathcal{L}_{jn}.$$

(2) with boundary condition $\psi_{jn}(0) = \psi_j$ and $\psi_{jn}(h) = \psi_n$ is solved by the standard ODE calculus:

$$\psi_{jn}(x) = \frac{\sin(kx)}{\sin(kh)}\psi_n + \frac{\sin(k(h-x))}{\sin(kh)}\psi_j + \frac{1}{k^2}\left(\frac{\sin(kx)}{\sin(kh)} - \frac{x}{h}\right)\varphi_n + \frac{1}{k^2}\left(\frac{\sin(k(h-x))}{\sin(kh)} - 1 + \frac{x}{h}\right)\varphi_j.$$

In particular,

$$\psi'_{jn}(j) = \psi'_{jn}(0) = \frac{k}{\sin(kh)}(\psi_n - \psi_j) + \frac{k(1-\cos(kh)))}{\sin(kh)}\psi_j + \frac{1}{k^2}\left(\frac{k}{\sin(kh)} - \frac{1}{h}\right)(\varphi_n - \varphi_j) + \frac{1-\cos(kh)}{k\sin(kh)}\varphi_j.$$

Substituting this for the boundary condition

$$\sum_{|j-n|=h} \psi'_{jn}(j) = hV_j\psi_j,$$

we have for any j,

$$-\frac{1}{h^2}\sum_{|n-j|=h}(\psi_n-\psi_j)+\frac{\sin(kh)}{kh}V_j\psi_j-k^2\nu\frac{1-\cos(kh)}{(kh)^2/2}\psi_j$$
$$=-\frac{\sin(kh)-kh}{(kh)^3}\sum_{|n-j|=h}(\varphi_n-\varphi_j)+\nu\frac{1-\cos(kh)}{(kh)^2/2}\varphi_j.$$

Let

$$(M_1\psi)_j=\nu^{-1}(\frac{\sin(kh)}{kh}-1)V_j\psi_j-k^2(\frac{1-\cos(kh)}{(kh)^2/2}-1)\psi_j,$$
$$(M_2\varphi)_j=-\nu^{-1}\frac{\sin(kh)-kh}{(kh)^3}\sum_{|n-j|=h}(\varphi_n-\varphi_j)+(\frac{1-\cos(kh)}{(kh)^2/2}-1)\varphi_j.$$

Then we have $(H_2-k^2+M_1)K\psi=(1+M_2)\varphi$.

Proof of Lemma 5

Taking into account the Taylor series expansion of $\frac{\sin(kh)}{kh}$, $\frac{1-\cos(kh)}{(kh)^2/2}$ and $\frac{\sin(kh)-kh}{(kh)^3}$ (all of them are $1+O(h^2)$), we see that

$$(H_2-k^2)^{-1}M_m = O(h^2), \quad m=1,2.$$

Thank you for your attention!

References:

- S. Nakamura, Y. T.: On a continuum limit of discrete Schrödinger operators on square lattice. Journal of Spectral Theory **11** (2021), no. 1, 355–367.
- P. Exner, S. Nakamura, Y. T.: Continuum limit of the lattice quantum graph Hamiltonian. Preprint arXiv:2202.06586.

Appendix: Proof of Lemma 3 (1)

We recall that $\mathcal{F}u(\xi) = \int_{\mathbb{R}^\nu} e^{-2\pi i x\cdot\xi} u(x)dx$, and denote the discrete Fourier transform

$$F : \mathcal{H}_2 \to \hat{\mathcal{H}}_2 = L^2(h^{-1}\mathbb{T}^\nu), \quad \mathbb{T} = \mathbb{R}/\mathbb{Z},$$

by

$$Fv(\zeta) = h^\nu \sum_{z\in h\mathbb{Z}^\nu} e^{-2\pi i z\cdot\zeta} v(z), \quad \zeta \in h^{-1}\mathbb{T}^\nu.$$

Then we have $H_0 = \mathcal{F}^* H_0(\cdot)\mathcal{F}$ and $H_{2,0} = F^* H_{2,0}(\cdot)F$, where

$$\begin{aligned} H_0(\xi) =& |2\pi\xi|^2, \quad \xi \in \mathbb{R}^\nu, \\ H_{2,0}(\zeta) =& 2h^{-2} \sum_{j=1}^{\nu} (1 - \cos(2\pi h\zeta_j)) \\ =& 4h^{-2} \sum_{j=1}^{\nu} \sin^2(\pi h\zeta_j), \quad \zeta \in h^{-1}\mathbb{T}^\nu. \end{aligned}$$

Momentum representation

Now we set $Q := F_h P_h \mathcal{F}^*$.

$$
\begin{array}{ccccccc}
\hat{\mathcal{H}}_2 & \xleftarrow{F} & \mathcal{H}_2 & \xrightarrow{(H_{2,0}-\mu)^{-1}} & \mathcal{H}_2 & \xleftarrow{F^*} & \hat{\mathcal{H}}_2 \\
Q \uparrow & & P \uparrow & & \downarrow P^* & & \downarrow Q^* \\
\hat{\mathcal{H}} = L^2(\mathbb{R}^\nu) & \xrightarrow[\mathcal{F}^*]{} & \mathcal{H} & \xrightarrow[(H_0-\mu)^{-1}]{} & \mathcal{H} & \xleftarrow[\mathcal{F}]{} & \hat{\mathcal{H}} = L^2(\mathbb{R}^\nu)
\end{array}
$$

Then we have

Lemma 6 (Representation of Q_h)

For $f \in \hat{\mathcal{H}}$ and $g \in \hat{\mathcal{H}}_2$,

$$
\begin{aligned}
Qf(\zeta) &= \sum_{n \in \mathbb{Z}^\nu} \overline{\hat{\varphi}(h\zeta + n)} f(\zeta + h^{-1}n), \quad \zeta \in h^{-1}\mathbb{T}^\nu, \\
Q^* g(\xi) &= \hat{\varphi}(h\xi)\tilde{g}(\xi), \quad \xi \in \mathbb{R}^\nu,
\end{aligned}
$$

where $\tilde{g}$ is the periodic extension of g on $\mathbb{R}^\nu$.

Proof of $\|(H_{2,0}-\mu)^{-1}P - P(H_0-\mu)^{-1}\|_{\mathcal{B}(\mathcal{H},\mathcal{H}_2)} \to 0$

Proof Since P^*: isometric and $\mathcal{F}$, F: unitary, we have

$$\begin{aligned}&\left\|(H_{2,0}-\mu)^{-1}P - P(H_0-\mu)^{-1}\right\| \\ &\quad = \left\|P^*(H_{2,0}-\mu)^{-1}P - P^*P(H_0-\mu)^{-1}\right\| \\ &\quad = \left\|Q^*(H_{2,0}(\cdot)-\mu)^{-1}Q - Q^*Q(H_0(\cdot)-\mu)^{-1}\right\|.\end{aligned}$$

Then we compute, for $f \in \mathcal{S}(\mathbb{R}^\nu)$,

$$\begin{aligned}&\left(Q_h^*(H_{0,h}(\cdot)-\mu)^{-1}Q_h - Q_h^*Q_h(H_0(\cdot)-\mu)^{-1}\right)f(\xi) \\ &\quad = \sum_{n\in\mathbb{Z}^\nu} \hat{\varphi}(h\xi)\overline{\hat{\varphi}(h\xi+n)}B_h(\xi+h^{-1}n)f(\xi+h^{-1}n),\end{aligned}$$

where $B_h(\xi) := (H_{0,h}(\xi)-\mu)^{-1} - (H_0(\xi)-\mu)^{-1}$.
In the following, we decompose RHS into $n=0$ part and $n \neq 0$ part.

Estimates of $\hat{\varphi}(h\xi)\overline{\hat{\varphi}(h\xi+n)}B_h(\xi+h^{-1}n)f(\xi+h^{-1}n)$

We compute $\||\hat{\varphi}(h\cdot)|^2 B_h f\|_{\hat{\mathcal{H}}} \leq \||\hat{\varphi}(h\cdot)|^2 B_h\|_{L^\infty}\|f\|_{\hat{\mathcal{H}}}$ and

$$\begin{aligned}
&\int |\hat{\varphi}(h\xi)\sum_{n\neq 0}\overline{\hat{\varphi}(h\xi+n)}g(\xi+h^{-1}n)|^2 d\xi \\
&\leq \int |\hat{\varphi}(h\xi)|^2 \sum_{n\neq 0}|\hat{\varphi}(h\xi+n)|^2 \sum_{n\neq 0}|g(\xi+h^{-1}n)|^2 d\xi \\
&= \int |\hat{\varphi}(h\xi)|^2(1-|\hat{\varphi}(h\xi)|^2)\sum_{n\neq 0}|g(\xi+h^{-1}n)|^2 d\xi \\
&= \int \sum_{n\neq 0}|\hat{\varphi}(h\xi-n)|^2(1-|\hat{\varphi}(h\xi-n)|^2)|g(\xi)|^2 d\xi \\
&= \int (1-|\hat{\varphi}(h\xi)|^2 - \sum_{n\neq 0}|\hat{\varphi}(h\xi-n)|^4)|g(\xi)|^2 d\xi,
\end{aligned}$$

with $g := B_h(\cdot)f(\cdot)$.

Estimates of $\hat{\varphi}(h\xi)\overline{\hat{\varphi}(h\xi + n)}B_h(\xi + h^{-1}n)f(\xi + h^{-1}n)$

Therefore the proof is completed if we check

$$\||\hat{\varphi}(h\cdot)|^2 B_h\|_{L^\infty} \to 0,$$
$$\|(1 - |\hat{\varphi}(h\cdot)|^2 - \sum_{n\neq 0} |\hat{\varphi}(h\cdot - n)|^4)^{\frac{1}{2}} B_h\|_{L^\infty} \to 0.$$

However this claim holds since

$$B_h(\xi) = (H_{2,0}(\xi) - \mu)^{-1} - (H_0(\xi) - \mu)^{-1} \to 0$$

as $h \to 0$ if $\mathrm{dist}(\xi, h^{-1}\mathbb{Z}^\nu \setminus \{0\}) \to \infty$, while $|\hat{\varphi}(0)| = 1$ implies both $|\hat{\varphi}(h\xi)| = (1 - \sum_{n\neq 0} |\hat{\varphi}(h\xi - n)|^2)^{\frac{1}{2}}$ and $1 - |\hat{\varphi}(h\xi)|^2 - \sum_{n\neq 0} |\hat{\varphi}(h\xi - n)|^4$ vanish on $h^{-1}\mathbb{Z}^\nu \setminus \{0\}$.