Horinzontal quasiconvex envelope in the Heisenberg group

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Outline

- Notions of convexity in the Heisenberg group $\mathbb{H}$;
- Quasiconvex envelope contruction through 1st-order non-local Hamilton-Jacobi equation;
- Obtaining horizontal convex hull of a given set via constructing the quasiconvex envelope of a function.

The Heisenberg group $\mathbb{H}$

The Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ endowed with non-commutative group multiplication for all $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$

$$(x_p, y_p, z_p) \cdot (x_q, y_q, z_q) = \left(x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p) \right).$$

The Lie Algebra of $\mathbb{H}$ is generated by the left-invariant vector fields

$$X_1(p) = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$$
$$X_2(p) = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$$

Note $X_1 X_2 \neq X_2 X_1$. The horizontal gradient of $u : \mathbb{H} \to \mathbb{R}$ are given by $\nabla_H u = (X_1 u, X_2 u)$.

Horizontal planes and various metrics in $\mathbb{H}$

The horizontal planes in $\mathbb{H}$ is the set of all planes

$$\mathbb{H}_p = Span\{X_1(p), X_2(p)\} = \{(x, y, z) : xy_p - yx_p + 2z - 2z_p = 0\},$$

and $\mathbb{H}_p = p \cdot \mathbb{H}_0 = p \cdot \{(x, y, 0) : x, y \in \mathbb{R}\}$.

A piecewise smooth curve $s \to \gamma(s) \in \mathbb{H}$ is called horizontal if its tangent vector $\gamma'(s) \in \mathbb{H}_p$ for every s where $\gamma'(s)$ exists. The Carnot-Caratheodory distance $d_{CC}(p, q)$ is given by:

$$d_{CC}(p, q) = \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \|\gamma'(s)\| \, ds.$$

The Korányi gauge on $\mathbb{H}$ is defined as $\|p\|_G = \left((x^2 + y^2)^2 + 16z^2\right)^{\frac{1}{4}}$ and the gauge metric is given by

$$d_G(p, q) = \|p^{-1} \cdot q\|_G$$

and is left-translation invariant.

Notions of convexity in the Heisenberg group

Definition (Lu-Manfredi-Stroffolini, '04) A function $u : \mathbb{H} \to \mathbb{R}$ is said to be horizontally convex (h-convex), if for every $p \in \mathbb{H}$ and $h \in \mathbb{H}_0 = \{(x, y, 0) : x, y \in \mathbb{R}\}$

$$u(p \cdot h^{-1}) + u(p \cdot h) \geq 2u(p).$$

An equivalent notion of horizontally convexity (called weakly *H*-convex) on Carnot group is given by [Danielli, Garofalo, and Nhieu, '03] and they also studied various notions of convexity in this work.

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Definition (Danielli-Garofalo-Nhieu, '03) We say, that a set $E \subset \mathbb{H}$ is h-convex if and only if for every $p \in E$ and $q \in \mathbb{H}_p \cap E$ the horizontal line segment $[p, q]$ joining p and q stays inside E.

h-convex sets are very different from Euclidean convex sets. Construction of h-convex hull of a given set is not obvious.

Geodetically convexity of sets in $\mathbb{H}$ is studied in [Monti-Rickly, '05]. Horizontally convex envelope of a given function is considered in [Liu-Z, 20].

Quasiconvex functions in $\mathbb{H}$

Definition (Sun-Yang, 05 and Calogero-Carcano-Pini, 08) Suppose that $\Omega \subset \mathbb{H}$ is h-convex. We say, that a function $u : \Omega \to \mathbb{R}$ is h-quasiconvex if for every $\lambda \in \mathbb{R}$ the sublevel set $\{w \in \Omega : u(w) \leq \lambda\}$ is h-convex. Equivalently, u is quasiconvex if and only if

$$u(w) \leq \max(u(p), u(q))$$

for every $p \in \Omega$, $q \in \mathbb{H}_p \cap \Omega$ and $w \in [p, q]$.

Defintion (h-quasiconvex envelope) Let Ω be an h-convex domain in $\mathbb{H}$ and $f : \Omega \to \mathbb{R}$ be a given function. We say, that $Q(f)$ is the h-quasiconvex envelope of f if it is the greatest h-quasiconvex function majorized by f, that is

$$Q(f)(p) := \sup\{g(p) : g \leq f \text{ and } g \text{ is h-quasiconvex}\}.$$

h-quasiconvex envelope

For an h-convex domain $\Omega \subset \mathbb{H}$ and $f : \Omega \to \mathbb{R}$ let us define

$$S[f](w) = \inf\left\{\max\{f(p), f(q)\} : w \in [p, q],\ p \in \Omega,\ q \in \Omega \cap \mathbb{H}_p\right\}.$$

- $\inf_\Omega f \leq S[f] \leq f$ in Ω;
- $S[f] = f$ in Ω provided f is already h-quasiconvex;
- Unlike in the Euclidean case that $Q_E(f) = S_E(f)$, we do not necessarily have $S_E(f)$ is a h-quasiconvex function in $\mathbb{H}$.

Example Let $f : \mathbb{H} \to \mathbb{R}$ be defined as $f(p) = |1 - z^2|$ for $p = (x, y, z) \in \mathbb{H}$. Then we can easily verify that $S(f)$ is not h-quasiconvex.

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Proposition Let Ω be an h-convex domain in $\mathbb{H}$. Suppose that f is bounded from below. Let $S^{n+1}[f](p) := S[S^n[f]](p)$ for $n = 1, 2, \ldots$. Then $S^n[f] \to Q(f)$ pointwise in Ω as $n \to \infty$.

First-order characterization of h-quasiconvexity in $\mathbb{H}$

Barron, Goebel and Jensen (Barron-Goebel-Jensen, 12) introduce a first-order characterization of quasiconvexity in $\mathbb{R}^n$ and use it to construct the quasiconvex envelope. Our work extend their results to the Heisenberg group.

Theorem Let $\Omega \subset \mathbb{H}$ be open and h-convex and $u : \Omega \to \mathbb{R}$ be upper semicontinuous. Then, u is h-quasiconvex if and only if whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p, there holds

$$u(\xi) < u(p) \Rightarrow \langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle \leq 0$$

for any $\xi \in \mathbb{H}_p \cap \Omega$.

The above theorem amounts to saying that $u \in USC(\Omega)$ is h-quasiconvex if

$$\sup\{\langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_p \cap \Omega,\ u(\xi) < u(p)\} \leq 0$$

holds in the viscosity sense. Hence, we define

$$H(p, u(p), \nabla_H u(p)) = \sup\left\{\langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in S_p(u)\right\},$$

where $S_p(u) = \{\xi \in \mathbb{H}_p \cap \Omega : u(\xi) < u(p)\}$.

Nonlocal Hamilton-Jacobi equation

Let f be a given function in a domain $\Omega \subset \mathbb{H}$. We focus on the study of the following nonlocal Hamilton-Jacobi equation:

$$u(p) + H(p, u(p), \nabla_H u(p)) = f(p) \quad \text{in } \Omega.$$

Definition (subsolution) Let $f \in USC(\Omega)$ be locally bounded in Ω. A locally bounded upper semicontinuous function $u : \Omega \to \mathbb{R}$ is called a subsolution of (1) if whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p,

$$u(p) + H(p, u(p), \nabla_H \varphi(p)) \leq f(p), \tag{1}$$

where $H(p, u(p), \nabla_H \varphi(p)) = \sup \left\{ \langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in S_p(u) \right\}$.

Definition (supersolution) Let $f \in LSC(\Omega)$ be locally bounded in Ω. A locally bounded lower semicontinuous function $u : \Omega \to \mathbb{R}$ is called a weak supersolution of (1) if whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a minimum at p,

$$u(p) + \sup \left\{ \langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle : \ \xi \in \tilde{S}_p(u) \right\} \geq f(p),$$

where $\tilde{S}_p(u) := \{\xi \in \mathbb{H}_p \cap \Omega : u(\xi) \leq u(p)\}$.

Properties of subsolutions

Let f be a given function in a domain $\Omega \subset \mathbb{H}$. A subsolution u satisfies in the viscosity sense that

$$u(p) + \sup\left\{\langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in S_p(u)\right\} \leq f(p) \quad \text{in } \Omega.$$

Remark If f is h-quasiconvex, then f is a subsolution to (1).

Any h-quasiconvex function majoried by f is a subsolution to (1).

Lemma Let $f \in USC(\Omega)$ be locally bounded in Ω. If $u \in USC(\Omega)$ is a subsolution of (1). Then $u \leq f$ in Ω.

Proposition (Maximum subsolution) Suppose that Ω is a domain in $\mathbb{H}$ and $f \in USC(\Omega)$. Let $\mathcal{A}$ be a family of subsolutions of (1). Let v be given by

$$v(p) = \sup\{u(p) : u \in \mathcal{A}\}, \quad p \in \Omega.$$

Then v^* is a subsolution of (1).

Quasiconvex envelope construction

Let Ω be a h-convex set. Let $f \in USC(\Omega)$. Assume there exists $\underline{f} \in USC(\Omega)$ h-quasiconvex such that

$$\underline{f} \leq f \quad \text{in } \Omega$$

Let $u_0 = f$. For $n = 1, 2, \ldots$, define u_n be the maximal subsolution of

$$u_n + H(p, u_n, \nabla_H u_n) = u_{n-1} \quad \text{in } \Omega \tag{2}$$

that is, $u_n = v_n^*$ in Ω, where

$$v_n = \sup\{u(p) : u \in USC(\Omega) \text{ is a subsolution of (2)}\}.$$

By the property of subsolution, one can also see that

$$\underline{f} \leq \ldots \leq u_n \leq u_{n-1} \leq \ldots \leq u_0 = f \quad \text{in } \Omega \text{ for } n = 1, 2, \ldots.$$

We can show that $u_n \to Q(f)$ in $\bar{\Omega}$ as $n \to \infty$.

Quasiconvex envelope construction via sequence of subsolutions

Proposition Suppose that Ω is a h-convex domain in $\mathbb{H}$. Let $u_0 = f \in USC(\Omega)$ and $u_n \in USC(\Omega)$ be a subsolution of (2). Then

$$u(p) = \limsup_{n\to\infty}{}^{*} u_n(p) = \lim_{k\to\infty} \sup \left\{ u_n(q) : q \in B_{\frac{1}{k}}(p),\ n \geq k \right\}, \quad p \in \Omega$$

is h-quasiconvex in Ω.

Recall $u_0 = f$ and u_n is a decreasing sequence satisfying

$$u_n + H(p, u_n, \nabla_H u_n) \leq u_{n-1} \quad \text{in } \Omega$$

- Note $\underline{f} \leq \ldots \leq u_n \leq u_{n-1} \leq \ldots \leq u_0 = f$ in Ω for $n = 1, 2, \ldots$.
- Since $\lim_{n\to\infty} u_n = \limsup_{n\to\infty}{}^{*} u_n$ when u_n is decreasing, it follows that $\lim_{n\to\infty} u_n \leq Q(f)$.
- On the other hand, $Q(f)$ is a subsolution to the equation and the maximality of u_n implies that $Q(f) \leq u_n$ for each n.

Remark We can also construct the h-quasiconvex envelope by taking u_n to be the solution of the above equation starting from $u_0 = f$ with boundary data f. It follows from the following results.

Uniqueness and Existence

Theorem (Comparison principle) Let Ω be a bounded domain in $\mathbb{H}$ and $f \in C(\Omega)$. Let $u \in USC(\bar{\Omega})$ be a subsolution, that is,

$$u(p) + \sup\left\{\langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \rangle : \ \xi \in S_p(u)\right\} \le f(p)$$

and $v \in LSC(\bar{\Omega})$ be a supersolution

$$v(q) + \sup\left\{\langle \nabla_H v(q), (q^{-1} \cdot \xi)_h \rangle : \xi \in \tilde{S}_q(v)\right\} \ge f(q).$$

If $u \le v \equiv c \quad$ on $\partial\Omega$ and $u \le c$ in Ω, then $u \le v$ in $\bar{\Omega}$.

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The main challenge compared to the Euclidean space lies in the requirement that points are in the intersection of the sublevel sets with the horizontal plane at p_ε and q_ε.

Theorem (Existence by Perron's method) Let $\Omega \subset \mathbb{H}$ be a bounded domain. Let $f \in C(\bar{\Omega})$ and satisfy $f = c$ on $\partial\Omega$ and $f \leq c$ in $\bar{\Omega}$ for some $c \in \mathbb{R}$. Assume that there exist a subsolution $\underline{u} \in C(\bar{\Omega})$ of (1) satisfying $\underline{u} \leq f$ in Ω and $\underline{u} = f$ on $\partial\Omega$. For any $p \in \bar{\Omega}$, let

$$U(p) = \sup\{u(p) : u \in USC(\Omega) \text{ is a subsolution of (1)}\}.$$

Then U^* is continuous in $\bar{\Omega}$ and is a solution of (1) satisfying $U^* = f$ on $\partial\Omega$.

Quasiconvex envelope construction via sequence of solutions

Let Ω be a bounded h-convex set. Let $f \in C(\bar{\Omega})$ and satisfy $f = c$ on $\partial\Omega$ and $f \leq c$ in $\bar{\Omega}$ for some $c \in \mathbb{R}$. Assume there exists $\underline{f} \in C(\bar{\Omega})$ h-quasiconvex such that

$$\underline{f} \leq f \quad \text{in } \bar{\Omega} \text{ and } \underline{f} = f \quad \text{on } \partial\Omega.$$

Let $u_0 = f$. By Existence and Uniqueness Theorem, for $n = 1, 2, \ldots$ we can find a unique solution u_n of

$$u_n + H(p, u_n, \nabla_H u_n) = u_{n-1} \quad \text{in } \Omega \tag{3}$$

satisfying

$$u_n = f \quad \text{on } \partial\Omega. \tag{4}$$

By the property of subsolution, one can also see that

$$\underline{f} \leq \ldots \leq u_n \leq u_{n-1} \leq \ldots \leq u_0 = f \quad \text{in } \Omega \text{ for } n = 1, 2, \ldots.$$

We can show that $u_n \to Q(f)$ in $\bar{\Omega}$ as $n \to \infty$.

h-convex hull through quasiconvex envelope

Definition (H-convex hull) For any set $E \subset \mathbb{H}$, the h-convex hull of E, denoted by $\mathrm{co}(E)$, is defined to be the smallest h-convex set in $\mathbb{H}$ containing E, i.e.,

$$\mathrm{co}(E) = \bigcap \{D \subset \mathbb{H} : D \text{ is h-convex and satisfies } E \subset D\}.$$

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1. For a given open set $E \subset \mathbb{H}$, we take a function $f \in C(\mathbb{H})$ such that
$$E = \{p \in \mathbb{H} : f(p) < 0\}.$$
2. We construct the h-quasiconvex envelope $Q(f)$.
3. The h-convex hull turns out to be the 0-sublevel set of $Q(f)$, that is,
$$\mathrm{co}(E) = \{p \in \mathbb{H} : Q(f)(p) < 0\}.$$

Theorem (Construct h-convex hull through quasiconvex envelope) Let $E \subset \mathbb{H}$ be an open set. Assume $f \in C(\mathbb{H})$ satisfies $E = \{p \in \mathbb{H} : f(p) < 0\}$. Let $Q(f)$ be the h-quasiconvex envelope of f. Then $Q(f) \in USC(\mathbb{H})$ and

$$\mathrm{co}(E) = \{p \in \mathbb{H} : Q(f) < 0\}.$$

Summary

- We studied the h-quasiconvex functions in the Heisenberg group $\mathbb{H}$ and gives two ways of constructing a quasiconvex envelope of a given function.
- One way is taking the limit of a sequence of functions achieved by applying the convexification operator.
- The other appraoch is based on one 1st-order characterization of quasiconvex functions. We can construct the quasiconvex envelope as the pointwise limit of the sequence of maximal subsolution.
- We also obtain the uniqueness and existence of the above non-local Hamilton-Jacobi equation. The quasiconvex envelope can be constructed as the pointwise limit of the sequence of solutions as well.
- Finally, we study the relation between *h*-convex hull and quasiconvex functions and apply this relation to find the *h*-convex hull of a given set.

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- Finally, we study the relation between h-convex hull and quasiconvex functions and apply this relation to find the h-convex hull of a given set.

Thank you!