Optimal Liouville-type theorems for system of parabolic inequalities

Anh Tuan DUONG
Hanoi University of Science and Technology
(This talk is based on a joint work with Quoc Hung Phan)

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where the exponents p and q are real numbers, I is an interval of $\mathbb{R}$.
We propose to study

(a) Liouville-type theorems for nonnegative solutions in whole space $\mathbb{R}^N \times \mathbb{R}$ and in $\mathbb{R}^N \times (0, \infty)$, provided that $p, q > 0$.

(b) Liouville-type theorems for positive solutions in $\mathbb{R}^N \times \mathbb{R}$ and in $\mathbb{R}^N \times (0, \infty)$ with real exponents p, q.

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Nonnegative solutions of (1)

The well-known Fujita result ensures the nonexistence of nontrivial nonnegative solution of problem (1) in $\mathbb{R}^N \times (0, \infty)$ under the condition $1 < p \leq \frac{N+2}{N}$, see [Fuj66][1] [MP01][2].

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When $p > \frac{N+2}{N}$, a nonnegative solution is, see [Kur15][3]

$$w(x,t) = \begin{cases} kt^{-\frac{1}{p-1}} e^{-\gamma \frac{1+|x|^2}{t}} & \text{if } t > 0, \ x \in \mathbb{R}^N \\ 0 & \text{if } t \leq 0, \ x \in \mathbb{R}^N \end{cases}.$$

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By using the rescaled test-function method, one can deduce the nonexistence of nontrivial nonnegative solutions of (2) in $\mathbb{R}^N \times \mathbb{R}$ in the range

$$p, q > 1 \quad \text{and} \quad \max\left\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right\} \geq N.$$ [1] [2]

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Under an additional assumption that solutions are spatially bounded, Escobedo and Herrero [3] proved a Liouville-type theorem for nonnegative solutions of parabolic system

$$\begin{cases} u_t - \Delta u = v^p, \\ v_t - \Delta v = u^q \end{cases} \tag{3}$$

in $\mathbb{R}^N \times (0, \infty)$,

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The problem

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has no positive classical solution in $\mathbb{R}^N \times \mathbb{R}$ *if and only if*

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The problem (2) *has no positive classical solution in* $\mathbb{R}^N \times \mathbb{R}$ *if and only if* (p, q) *is in one of the following ranges*

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Content

Proof of Theorem 1

As mentioned before, our contribution is the proof of nonexistence result for

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in the case $p < 1$. Remark that, unlike the case $p > 1$ where one can use the test-function method, the case $p < 1$ requires another approach. We deal with this case by using a suitable change of variable and developing the argument of maximum principle inspired by Cheng, Huang and Li [1]

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Set $z := w^{-1}$, then (1) becomes

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Proof of Theorem 2: Existence of nonnegative solutions

When $pq < 1$, a nontrivial nonnegative solution (u, v) in $\mathbb{R}^N \times \mathbb{R}$ of the following form

$$(u, v) = \begin{cases} (At^{\alpha}, Bt^{\beta}) & \text{if } t > 0, x \in \mathbb{R}^N, \\ (0, 0) & \text{if } t \le 0, x \in \mathbb{R}^N \end{cases}.$$

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$$(u, v) = \begin{cases} (At^{\alpha}, Bt^{\beta}) & \text{if } t > 0, x \in \mathbb{R}^N, \\ (0, 0) & \text{if } t \leq 0, x \in \mathbb{R}^N \end{cases}.$$

When $pq = 1$, a positive solution is of the form

$$(u, v) = \left(\frac{1}{p\beta} e^{p\beta t},\ e^{\beta t}\right), \qquad \text{where } \beta = p^{-\frac{q}{q+1}}.$$

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We prove the nonexistence result by contradiction.

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Content

We address a similar question on the optimal Liouville-type theorems for the positive or nonnegative solutions of the fractional parabolic equation

$$u_t + (-\Delta)^s u \geq u^p \text{ in } \mathbb{R}^N \times I$$

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Here $c_{N,s}$ is the normalization constant and $P.V.$ stands for the Cauchy principle value. This operator is also defined by using the Fourier transform

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We also obtain similar results as in the case of Laplace operator [DN21][1]

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Assume that $p > 0$. Then the equation has no nontrivial nonnegative solution in $\mathbb{R}^N \times \mathbb{R}$ if and only if

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Theorem 6

The system has no positive solution in $\mathbb{R}^N \times \mathbb{R}$ *if* (p, q) *is in one of the following ranges*

- $p \le 0$ *or* $q \le 0$.
- $p, q > 0$ *and* $pq < 1$.
- $p, q > 0,\ pq > 1$ *and* $\max\left\{\frac{2s(p+1)}{pq-1}, \frac{2s(q+1)}{pq-1}\right\} > N$.

In addition, the system has positive solutions in $\mathbb{R}^N \times \mathbb{R}$ *if*

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Notice that the critical case is left open. [KO17][1]

[1] Kakehi, Tomoyuki; Oshita, Yoshihito; Blowup and global existence of a solution to a semilinear reaction-diffusion system with the fractional Laplacian. Math. J. Okayama Univ. 59 (2017), [2016 on cover], 175–218.

THANK YOU VERY MUCH FOR YOUR ATTENTION