Spontaneous mass generation and chiral symmetry breaking in a lattice Nambu-Jona-Lasinio model

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QCD Theory

- QCD Lagrangian: $\psi = (u, d, s, \dots) = N_f$-quark, $t^a = \mathrm{SU}(3)$-color matrix

$$\mathcal{L}_{\mathrm{QCD}} := \underbrace{\bar{\psi}(i\gamma^\mu D_\mu - m)\psi}_{\text{quark}} - \underbrace{\mathrm{tr}\, F^a_{\mu\nu} F^{\mu\nu}_a / 4}_{\text{Yang-Mills}}, \quad D_\mu := \partial_\mu - igt^a \underbrace{A^a_\mu}_{\text{gluon}}$$

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \quad (a, b, c = 1, 2, \dots, 8)$$

- Chiral transformation: For $U_{R/L} \in \mathrm{SU}(N_f)$

$$\psi_L = \frac{1 - \gamma_5}{2}\psi \to U_L \psi_L, \quad \psi_R = \frac{1 + \gamma_5}{2}\psi \to U_R \psi_R$$

$\mathcal{L}_{\mathrm{QCD}}$ is invariant under the chiral transformation when $m = 0$.

Effective (Low-Energy) Theory

A simplest effective theory: Nambu-Jona-Lasinio (NJL) model.

$$\mathcal{L}_{\mathrm{NJL}} := \bar{\psi} i\gamma^\mu \partial_\mu \psi + g \underbrace{\left[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5 \psi)^2\right]}_{\text{4-fermion}} \quad \text{(without gluon)}$$

$$= \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R + g\bar{\psi}_R \psi_L \bar{\psi}_L \psi_R$$

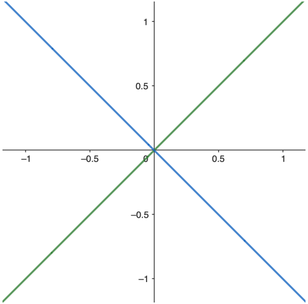
$\mathcal{L}_{\mathrm{NJL}}$ has the chiral symmetry.
In physics literature, it shows that the dynamical mass $M \sim \langle \bar{\psi}\psi \rangle \neq 0$ when $g > g_c \rightarrow$ Spontaneous breakdown of chiral symmetry $\simeq$ quark mass generation.

Main issue in this talk

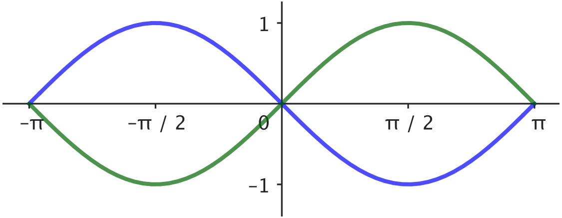
Proof of mass generation in a mathematically rigorous way.

- Lattice model: Some known results...

Lattice Dirac and Doubling Problem



(a) dispersion relation on $-i\partial_x\psi$

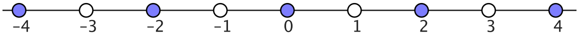


(b) dispersion relation on $\mathbb{Z}$

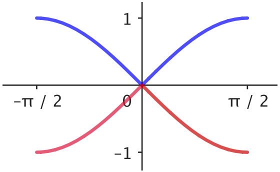
Discrete Dirac operator is $p_a\psi = \dfrac{1}{2ai}[\psi(x+a) - \psi(x-a)]$ and $\omega = \sin(ka)$ $(-\pi \le k \le \pi)$.
For dimensions ν, lattice fermions have 2^ν degeneracy.
Cause: The difference is $2a$ for lattice spacing a.

Staggered Fermion: Kogut-Susskind Formalism

Let ψ_1 for even points and ψ_2 for odd. Two Eq: For $a \mapsto 2a$,
$p_a\psi_1 = -i[\psi_1(x+a) - \psi_1(x)]/a$, $p_a\psi_2 = -i[\psi_2(x) - \psi_2(x-a)]/a$.
$\therefore \widehat{(p_a\psi_1)}(k) = e^{ika/2}\sin(ka/2)/(a/2)$ and $\widehat{(p_a\psi_2)}(k) = e^{-ika/2}\sin(ka/2)/(a/2)$. (Fig.)
For Dirac matrices $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\bar{\psi}\gamma_5\psi = \psi^\dagger\gamma_0\gamma_5\psi = \psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1$.
Hamiltonian is obtain by $H = i\sum_x \bar{\psi}\gamma_5 p_0\psi = i\sum_x (-1)^x[\varphi^\dagger(x)\varphi(x+1) - \text{h.c.}]$ for a one component field φ.



(a) Even and Odd points



(b) Even = Blue, Red = Odd

Staggered Fermion: Lattice NJL

We consider the staggered fermion + four-fermion interaction (NJL) in $\Lambda = [-L+1, L]^\nu$.
Two formalism:

- Lagrangian: $\psi, \bar{\psi}$: Grassmann
  When $m = 0$, the action is invariant under

$$\psi(x) \to e^{i\alpha\varepsilon(x)}\psi(x), \quad \bar{\psi}(x) \to e^{i\alpha\varepsilon(x)}\bar{\psi}(x), \quad \varepsilon(x) = (-1)^{\sum_{\mu=1}^{\nu} x_\mu}, \quad \alpha \in \mathbb{R}$$

Theorem (Salmhofer-Seiler '91)

In $\Lambda \to \mathbb{Z}^\nu$, the mass generation $\langle\bar{\psi}\psi\rangle \neq 0$ occurs when $\nu \geq 4$.

- Hamiltonian: ψ : Fermion operator

Kogut-Susskind Hamiltonian

$\Lambda = [-L+1, L]^\nu$, $\{\psi^\dagger(x), \psi(y)\} = \delta_{x,y}$, $\{\psi(x), \psi(y)\} = 0$, $a =$ lattice spacing.

$$H(m) := \frac{i}{a} \sum_{x\in\Lambda} \sum_{\mu} (-1)^{\theta_\mu(x)} [\psi^\dagger(x)\psi(x+e_\mu) - \psi^\dagger(x+e_\mu)\psi(x)]$$
$$+ m \sum_x (-1)^{\sum_\mu x^{(\mu)}} \rho(x) + g \sum_{x,\mu} \rho(x)\rho(x+e_\mu), \quad \rho(x) = \psi^\dagger(x)\psi(x) - \frac{1}{2}$$

$$\theta_\mu(x) := \begin{cases} x^{(1)} + \cdots + x^{(\mu-1)} & \text{for } x^{(\mu)} \neq L \\ x^{(1)} + \cdots + x^{(\mu-1)} + 1 & \text{for } x^{(\mu)} = L, \end{cases} \quad x^{(0)} := 0$$

- $H(0)$ does not have usual continuous chiral symmetry.
- $H(0)$ is invariant under a certain discrete chiral transformation.

Particle-Hole Symmetry and Mass Generation

$$U_{\mathrm{PH}} := \prod_{x\in\Lambda}\prod_{y\neq x}(-1)^{\psi^\dagger(y)\psi(y)}(\psi^\dagger(x)+\psi(x)) \quad \Rightarrow \quad U_{\mathrm{PH}}^\dagger\psi(x)U_{\mathrm{PH}} = \psi^\dagger(x)$$

$$\therefore \quad U_{\mathrm{PH}}^\dagger H(0)U_{\mathrm{PH}} = H(0) \Rightarrow \langle\rho(x)\rangle_{\beta,m=0} = 0, \quad \langle A\rangle_{\beta,m} := Z_{\beta,m}^{-1}\,\mathrm{tr}(Ae^{-\beta H(m)})$$

Theorem (Mass Generation in Lattice NJL)

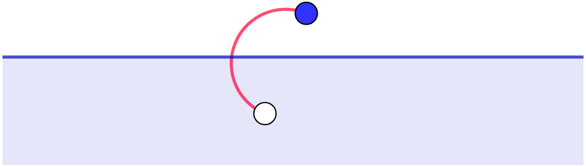
Assume $\nu \geq 3$*. For* $g_c \leq ag$, $\beta \geq \beta_c$ *we can show that*

$$\lim_{m\to 0}\lim_{\Lambda\to\mathbb{Z}^\nu}\langle\mathcal{O}_\Lambda\rangle_{\beta,m} := \lim_{m\to 0}\lim_{\Lambda\to\mathbb{Z}^\nu}\left\langle\sum_x(-1)^{\sum_\mu x^{(\mu)}}\rho(x)\right\rangle_{\beta,m} \neq 0.$$

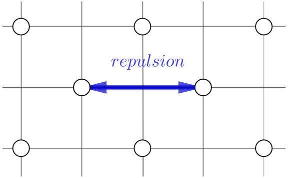
When $\nu \geq 2$ *&* $g_c \leq ag$, $\lim_{m\to 0}\lim_{\Lambda\to\mathbb{Z}^\nu}\lim_{\beta\to\infty}\langle\mathcal{O}_\Lambda\rangle_{\beta,m} \neq 0$.

- Breaking the discrete symmetry

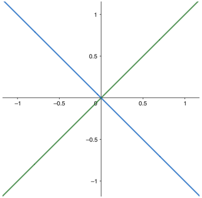
Image of Phenomena



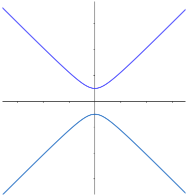
(a) Dirac sea



(b) Checker boards pattern



(c) dispersion relation on massless case



(d) massive case

- This is essentially the same as Mott (Metal-insulator) transition.

Tools of Proof

1. Fermion Reflection Positivity (RP)
2. Gaussian Domination (GD) Bound
3. Infrared (IR) Bound

These imply the existence of Long-Range Order (LRO) i.e., Phase transition.
LRO implies SSB (Koma-Tasaki, 1993).
IR bound: Let Λ^* be the dual lattice of Λ, and

$$\hat{\rho}_p := |\Lambda|^{-1/2} \sum_{x \in \Lambda} \rho(x) e^{ip \cdot x}, \quad p \in \Lambda^*.$$

$$(A, B) := Z^{-1} \int_0^1 ds \, \mathrm{tr}(e^{-s\beta H(m)} A e^{-(1-s)\beta H(m)} B) \quad \text{(Duhamel 2-point func.)}$$

$$(\hat{\rho}_p, \hat{\rho}_{-p}) \le \frac{1}{2\beta g E_{p+Q}} \quad \text{(IR bound)}$$

Here $2E_p := \sum_i [1 - \cos(p^i)]$ and $Q := (\pi, \ldots, \pi)$.

Infrared Bound to Inequalities

$$\langle \hat\rho_p \hat\rho_{-p} + \hat\rho_{-p}\hat\rho_p \rangle \le \sqrt{\frac{\nu}{agE_{p+Q}}} + \frac{1}{\beta g E_{p+Q}}$$

$$\therefore\ |\Lambda|^{-1} \sum_{p\in\Lambda^*} \langle \hat\rho_p \hat\rho_{-p} + \hat\rho_{-p}\hat\rho_p \rangle \le \sum_{p\neq Q} \left(\sqrt{\frac{\nu}{agE_{p+Q}}} + \frac{1}{\beta g E_{p+Q}} \right) + \underbrace{\frac{2\langle \hat\rho_Q \hat\rho_Q \rangle}{|\Lambda|}}_{=:2m_{\mathrm{LRO}}^{(\Lambda)}}$$

Here $m_{\mathrm{LRO}}^{(\Lambda)} = |\Lambda|^{-1}\sqrt{\langle \mathcal{O}_\Lambda^2 \rangle}$ = LRO parameter. By $\rho(x)^2 = 1/4$ & Plancherel,

$$\sum_{p\in\Lambda^*} \langle \hat\rho_p \hat\rho_{-p} + \hat\rho_{-p}\hat\rho_p \rangle = \frac{|\Lambda|}{2}$$

LRO in Finite & Zero Temperature

Letting $\Lambda \to \mathbb{Z}^\nu$

$$\frac{1}{4} \le \sqrt{\frac{\nu}{ag}} \underbrace{\frac{1}{(2\pi)^\nu} \int_{[-\pi,\pi]^\nu} \frac{dp}{\sqrt{E_p}}}_{=:J_\nu} + \frac{1}{\beta g} \underbrace{\frac{1}{(2\pi)^\nu} \int_{[-\pi,\pi]^\nu} \frac{dp}{E_p}}_{=:I_\nu} + m_{\mathrm{LRO}}^2$$

Since $E_p \sim p^2 + O(p^4)$, $I_\nu, J_\nu < \infty$ when $\nu \ge 3$.
Hence LRO ($m_{\mathrm{LRO}} > 0$) if ag & β large.
Taking $\beta \to \infty$ before $\Lambda \to \mathbb{Z}^\nu$:

$$\frac{1}{4} \le \sqrt{\frac{\nu}{ag}} J_\nu + m_{\mathrm{GSLRO}}^2.$$

$J_\nu < \infty$ in $\nu \ge 2$. Hence LRO if ag large.

Gaussian Domination Bound

Note

$$H_{\text{int}} = g \sum_{x\in\Lambda} \sum_{\mu} \rho(x)\rho(x+e_\mu) = \frac{g}{2}\sum_{x,\mu}[\rho(x)+\rho(x+e_\mu)]^2 + \text{const.}$$

Let $h = (h^{(1)}, \ldots, h^{(\nu)})$: $\mathbb{C}$-valued functions and

$$H_{\text{int}}(h) := \frac{g}{2}\sum_{x,\mu}\left[\rho(x)+\rho(x+e_\mu)+h^{(\mu)}(x)\right]^2 + \text{const.}$$

$$H(m,h) := H_{\mathsf{free}} + m\mathcal{O}_\Lambda + H_{\text{int}}(h), \quad Z(h) := \operatorname{tr}\exp(-\beta H(m,h))$$

GD bound: For all h: $Z(h) \le Z(0)$ holds.

By GD, $d^2Z(\varepsilon h)/d\varepsilon^2|_{\varepsilon=0} \le 0$. Then for $\partial_j h^{(\mu)}(x) := h^{(\mu)}(x) - h^{(\mu)}(x-e_j)$

$$\left(\rho\left[\overline{\sum_\mu \partial_\mu h^{(\mu)}}\right], \rho\left[\sum_\mu \partial_\mu h^{(\mu)}\right]\right) \le \frac{1}{\beta g}\sum_{\mu=1}^{\nu}\sum_{x\in\Lambda}|h^{(\mu)}(x)|^2, \quad \rho[f] := \sum_x \rho(x)f(x)$$

Taking $h^{(\mu)}(x) = |\Lambda|^{-1/2}(e^{ip\cdot(x+e_\mu)} - e^{ip\cdot x})$, we have IR bound.

Reflection Positivity

Divide $\Lambda = \Lambda_- \cup \Lambda_+$ and take r s.t. $r(\Lambda_\pm) = \Lambda_\mp$.
E.g. $\Lambda_- = \{x\colon -L+1 \le x^{(1)} \le 0\}$, $r(x^{(1)}) = -x^{(1)}$.
Define $\mathcal{A}_\pm =$ algebra of $\psi(x), \psi^\dagger(y)$, $x, y \in \Lambda_\pm$, and anti-linear $\vartheta : \mathcal{A}_\pm \to \mathcal{A}_\mp$ s.t. $(\vartheta\psi)(x) = \psi(\vartheta(x))$, $\vartheta(AB) = \vartheta(A)\vartheta(B)$ for $A, B \in \mathcal{A}_\pm$, etc.
In usual, RP is

$$\mathrm{tr}(A\vartheta(A)e^{-H}) \ge 0, \quad A \in \mathcal{A}_\pm$$

if $H = H_+ + H_- + H_0$, with $H_\pm \in \mathcal{A}_\pm$, $H_0 = \sum_i A_i\vartheta(A_i)$, $A_i \in \mathcal{A}_\pm$.
This fails for fermions.
Fermion RP: $\mathrm{tr}(A\theta(A)) \ge 0$ holds for $A \in \mathcal{A}_\pm$.
Using the Lie-Trotter formula: $e^{A+B+C} = \lim_{n\to\infty}[(1+A/n)e^{B/n}e^{C/n}]^n$,
$\mathrm{tr}\, e^{-\beta H(m,h)} = \lim_n \mathrm{tr}(\alpha_n^n)$,

$$\alpha_n := \left(1 - \frac{\beta}{n}H_0^{\text{free}}\right) \prod_{\substack{x\in\Lambda:\\ x^{(1)}=0,L}} e^{-\frac{\beta g}{2n}(\rho(x)-\rho(x+e_1)+h^{(1)}(x))^2} e^{-\frac{\beta}{n}H^-(m,h)} e^{-\frac{\beta}{n}H^+(m,h)}$$

Reflection Positivity to GD

$\exists\mu$, $\exists W_\pm \in \mathcal{A}_\pm$ we can write

$$\mathrm{tr}(\alpha_n^n) = \int d\mu(k) \sum_j \mathrm{tr}[W_-(j,k)W_+(j,k)] \prod_{x^{(1)}=0,L} e^{i\Theta(k)\sqrt{\frac{\beta}{2n}}h^{(1)}(x)}$$

$\mathrm{tr}(A\theta(A)) \geq 0 \Rightarrow (A,B) := \mathrm{tr}(A\theta(B))$ for $A, B \in \mathcal{A}_\pm$ is an inner product. By Schwarz

$$|\mathrm{tr}(\alpha_n^n)|^2 \leq \int d\mu(k) \sum_j \mathrm{tr}\left[W_-(j)\vartheta(W_-(j))\right] \int d\mu(k) \sum_j \mathrm{tr}\left[W_+(j)\vartheta(W_+(j))\right].$$

Undoing, $Z(h)^2 \leq \mathrm{tr}\, e^{-\beta(H_-(m,h)+\vartheta(H_-(m,h))+H_0(h)))} \times \mathrm{tr}\, e^{-\beta(H_+(m,h)+\vartheta(H_+(m,h))+H_0(h))}$

Then $Z(h)^2 \leq Z(h_+)Z(h_-)$, for $h_\pm := \begin{cases} h(x) & (x \in \Lambda_\pm \backslash \{x^{(1)} = 0, L\}) \\ h(\vartheta(x)) & (x \in \Lambda_\mp \backslash \{x^{(1)} = 0, L\}) \\ 0 & \text{else.} \end{cases}$

Proof of Gausian Domination

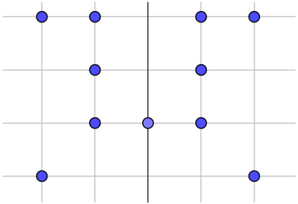


Figure: Divide $x^{(1)} = L$ plane.

Let h_0 to be a maximizer that contains the maximal number of zeros. We claim that $h_0^{(i)}(x) = 0$ for all i, x. Suppose that $h_0^{(1)}(L, x^{(2)}, \ldots, x^{(\nu)}) \neq 0$. Then, since $Z(h_0)^2 \leq Z(h_+)Z(h_-)$, $h_\pm$ obtained from h_0 are also maximizers containing more zero than h_0. Hence $h_0 \equiv 0$. QED

Rem: Reflection Positivity for any hyperplane is crucial.

Summary and Other Problems

Kogut-Susskind (KS) Hamiltonian: Lattice version of NJL model.

- KS Hamiltonian does not have chiral symmetry.
- Mass term $\langle \rho(x) \rangle = 0$ when $m = 0$ in KS Hamiltonian.
- For a large coupling constant, we can show the mass generation in the infinite-volume limit.
- Fermion reflection positivity is crucial.
- If we can take the continuum limit, the chiral symmetry is broken.

I don't know

- KS Hamiltonian $\leftrightarrow$ Grassmann theory?
- Proof of mass gap (spectral gap).
- Existence of the continuum limit.