The Poisson problems on domains with holes concentrated at subsets of a domain

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Homogenization problem

Let Ω :bounded,open$\subset \mathbb{R}^d, d \geq 3, \partial\Omega : C^2$,
$f \in L^2(\Omega)$, T_ε :closed set$\to \emptyset$ in "some sense".
$\Omega_\varepsilon := \Omega \setminus T_\varepsilon$.
Consider

$$u^\varepsilon \in H_0^1(\Omega_\varepsilon) \text{ with } -\Delta u^\varepsilon = f.$$

Regard $H_0^1(\Omega_\varepsilon) \subset H_0^1(\Omega)$ by 0 extension.
Homogenization problem:
Find μ satisfying $\|u^\varepsilon - u\|_{L^2(\Omega)} \to 0$ where

$$u \in H_0^1(\Omega) \text{ with } (-\Delta + \mu)u = f.$$

$\mu = 0$ i.e. $-\Delta u = f$

$S_d := \int_{\partial B(0,1)} dS$.

Definition (Newtonian capacity)

$\forall K$:*compact*$\subset \mathbb{R}^d$,
$m_{1K} = \{\nu$: *Borel measure on* $K, \nu(K) = 1\}$.
$\mathrm{cap}(K)^{-1} = \inf_{\nu \in m_{1K}} \iint \frac{|x-y|^{2-d}}{(d-2)S_d} d\nu(x) d\nu(y)$.

Example 1

$\mathrm{cap}(\overline{B(x,R)}) = S_d(d-2)R^{d-2}$.

Example 2 ([Rauch and Taylor, 1975])

$\overline{\bigcup_\varepsilon T_\varepsilon} \subset \Omega, \mathrm{cap}(T_\varepsilon) \to 0 \Rightarrow \mu = 0$.

Weak solution to $(-\Delta + \mu)u = f$

Let $\mu \in W^{-1,\infty}(\Omega) = W_0^{1,1}(\Omega)^*, \mu \geq 0$ in $\mathcal{D}'(\Omega)$ and

$$a(u, v) = (\nabla u, \nabla v)_2 + \langle \mu, u\overline{v} \rangle_{W^{-1,\infty}(\Omega)} \ (u, v \in H_0^1(\Omega))$$

be form on $L^2(\Omega)$.$\|\cdot\|_{H^1(\Omega)} \sim \|\cdot\|_a$. $\therefore a$:closed form. Riesz theorem gives $\exists^1 u \in H_0^1(\Omega)$,

$$\forall g \in H_0^1(\Omega), \ a(u, g) = (f, g)_2. \qquad (\mu\text{Poi})$$

i.e.

$$L_\mu u = f$$

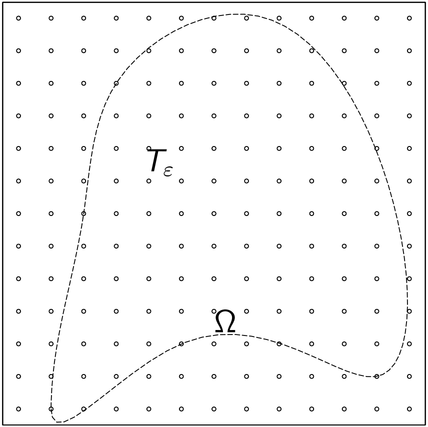
where $L_\mu = L_\mu^*$ is operator defined by form a.

$\mu \neq 0$ [Cioranescu and Murat, 1997]

Example 3 (Periodic balls)

$T_\varepsilon = \bigcup_{i \in 2\varepsilon\mathbb{Z}^d} \overline{B(i, \varepsilon^{\frac{d}{d-2}})} \Rightarrow \mu = \frac{S_d(d-2)}{2^d}$.

It is known that origin of μ is $\lim_\varepsilon \frac{\mathrm{cap}(\overline{B(0,\varepsilon^{\frac{d}{d-2}})})}{|(-\varepsilon,\varepsilon]^d|}$ (for ex, [Khrabustovskyi and Post, 2017]).



$\mu \neq 0$ [Cioranescu and Murat, 1997]

Example 4 (Periodic balls on $\mathbb{R}^{d-1} \times \{0\}$)

$T_\varepsilon = \bigcup_{i \in 2\varepsilon\mathbb{Z}^{d-1} \times \{0\}} \overline{B(i, \varepsilon^{\frac{d-1}{d-2}})} \Rightarrow \mu = \frac{S_d(d-2)}{2^{d-1}} \delta_{\mathbb{R}^{d-1} \times \{0\}}$.

Example 5 (Periodic cylinders)

$d = 3$, $T_\varepsilon = \bigcup_{i \in 2\varepsilon\mathbb{Z}^2 \times \{0\}} \overline{B_{\mathbb{R}^2}(i, e^{-\varepsilon^{-2}})} \times \mathbb{R} \Rightarrow \mu = \frac{3\pi}{2}$.

Framework

Example 3,Example 4,Example 5 are shown by Theorem 1.

Assumption 1

$\exists w^\varepsilon \in H^1(\Omega) \cap H_0^1(T_\varepsilon^c) \to 1$ *weakly in* $H^1(\Omega)$,
$\mu \in W^{-1,\infty}(\Omega)$,
$v^\varepsilon \in H_0^1(\Omega_\varepsilon) \to v$ *weakly in* $H_0^1(\Omega)$
$\quad \Rightarrow \langle -\Delta w^\varepsilon, v^\varepsilon \rangle_{H^{-1}(\Omega)} \to \langle \mu, v \rangle$.

Theorem 1 ([Cioranescu and Murat, 1997])

Under Assumption 1, we have $\mu \geq 0$ *in* $\mathcal{D}'(\Omega)$ *and*

$$u^\varepsilon \to L_\mu^{-1} f \text{ weakly in } H_0^1(\Omega).$$

Preliminaries for Theorem 1

Lemma 1

$\left\| L_\mu^{-1} \right\|_{L^2(\Omega)\to H^1(\Omega)} \leq (2+\mathrm{diam}(\Omega)^2)^2 =: p_\Omega.$

Proof.

$u := L_\mu^{-1} f$. $\|u\|_2^2 \leq c\|\nabla u\|_2^2 \overset{(\mu\mathrm{Poi})}{\leq} ca(u,u) \leq c\|f\|_2\|u\|_2$.
$\therefore \|u\|_2 \leq c\|f\|$. $\therefore c\|\nabla u\|_2^2 \leq c^2\|f\|_2^2$. □

Lemma 2

Let X : *norm space,* $\sup_n \|x_n\| < \infty$, *any subsequence of* x_n *converge weakly have same limit* x.
Then, $x_n \to x$ *weakly.*

Proof of Theorem 1

$\forall g \in \mathcal{D}(\Omega), \langle -\Delta w^\varepsilon, g w^\varepsilon \rangle$
$= \int |\nabla w^\varepsilon|^2 g dx + \int (w^\varepsilon - 1) \nabla w^\varepsilon \cdot \nabla g dx + (\nabla w^\varepsilon, \nabla g)_2.$
$\langle \mu, 1g \rangle = \lim_\varepsilon \int |\nabla w^\varepsilon|^2 g dx$ by Rellich theorem, etc. $\therefore \mu \geq 0$.
$\|u^\varepsilon\|_{H^1(\Omega)} \overset{Lemma\ 1}{\leq} p_{\Omega_\varepsilon} \|f\|_2 \leq p_\Omega \|f\|_2.$
For any subsequence u^{ε_n} converge weakly, and limit u,

$$(\nabla u^{\varepsilon_n}, \nabla (w^{\varepsilon_n} g))_2 = (f, w^{\varepsilon_n} g)_2$$

$\rightarrow (\mu\text{Poi})$ by Assumption 1.
$\therefore u = L_\mu^{-1} f$. Use Lemma 2.

Application to heat, Schrödinger equation

Theorem 2

Let Δ_ε^D : Dirichlet Laplacian on $L^2(\Omega_\varepsilon)$, P_ε : restriction to Ω_ε. Under Assumption 1,

$$\forall f \in L^2(\Omega),\ \left\| P_\varepsilon^* g(-\Delta_\varepsilon^D) P_\varepsilon f - g(L_\mu) f \right\|_2 \to 0.$$

for $g(\lambda) = e^{-it\lambda}$ $(t \in \mathbb{R}), e^{-t\lambda}$ $(t \geq 0)$.

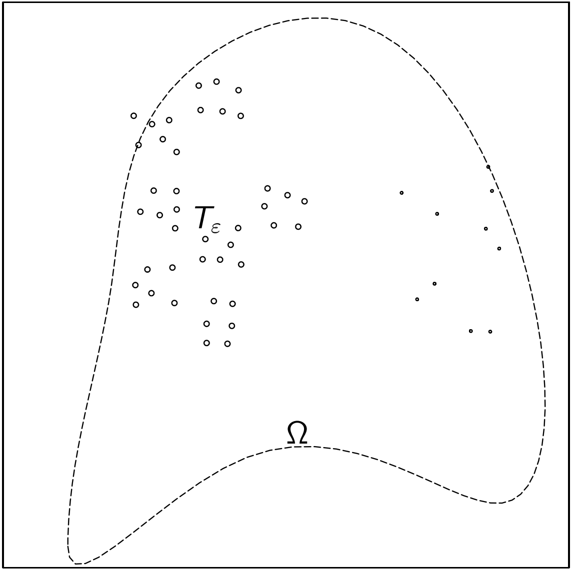
Proof.

If $g(\lambda) = \lambda^{-1}$, it is Theorem 1. Analogy of [Reed and Simon, 1980, VIII.20] shows this. □

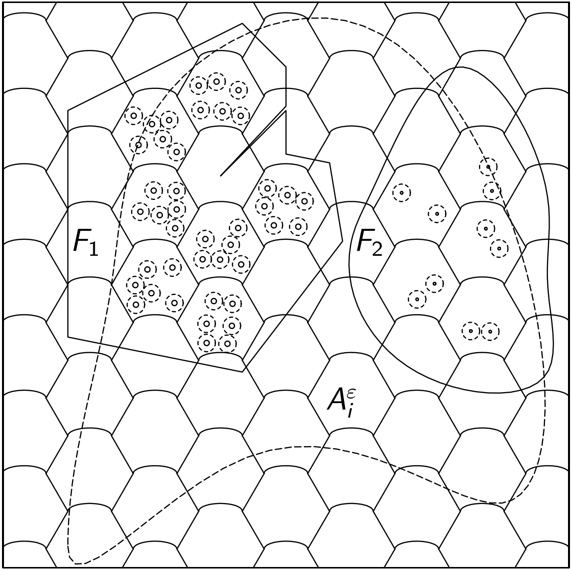
See [Dondl et al., 2017] about Norm Convergence, spectrum, Neumann, Robin Laplacian.

Concentrated holes (I. arXiv:2208.04557)

We consider T_ε concentrated at subset of $\mathbb{R}^d$.

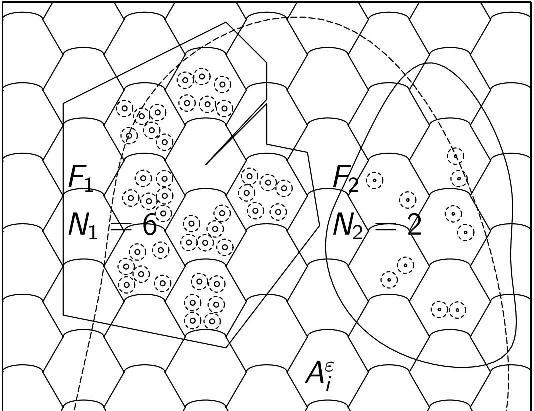


Construction of T_ε



Construction of T_ε

$\mathcal{J} := \{E \subset \mathbb{R}^d \mid |\mathring{E}| = |\overline{E}|\}$. $\mathcal{J}_b := \{E \in \mathcal{J} : \text{bounded}\}$.
$\{F_k\}_{k=1}^m$:disjoint $\subset \mathcal{J}$. $\{N_k\}_{k=1}^m \subset \mathbb{N}$. A :measurable,
$\#\Lambda = \#\mathbb{N}, \mathbb{R}^d = \bigsqcup_{i\in\Lambda}(A+i)$.
A can contain $\max_{k\le m} N_k$ balls with radius $C > 0$.
For $\varepsilon > 0, i \in \Lambda, A_i^\varepsilon := \varepsilon(A+i)$.
For $E \subset \mathbb{R}^d, \Lambda_\varepsilon^-(E) := \{i \in \Lambda \mid A_i^\varepsilon \subset E\}$.



Result

For $i \in \Lambda_\varepsilon^-(F_k)$. $\{x_{i,j}^\varepsilon\} \subset \mathbb{R}^d$ satisfy
$B^{i,\varepsilon} := \bigsqcup_{j \leq N_k} B(x_{i,j}^\varepsilon, C\varepsilon) \subset A_i^\varepsilon$. $T^{i,\varepsilon} := \bigsqcup_{j \leq N_k} \overline{B(x_{i,j}^\varepsilon, a_{\varepsilon,k})}$.

$$\varepsilon^{-d}(a_{\varepsilon,k})^{d-2} \to \mu_k \in [0, \infty). \qquad \text{(radii)}$$

$T_{\varepsilon,k} := \bigsqcup_{i \in \Lambda_\varepsilon^-(F_k)} T^{i,\varepsilon}$. $T_\varepsilon := \bigsqcup_{k \leq m} T_{\varepsilon,k}$.

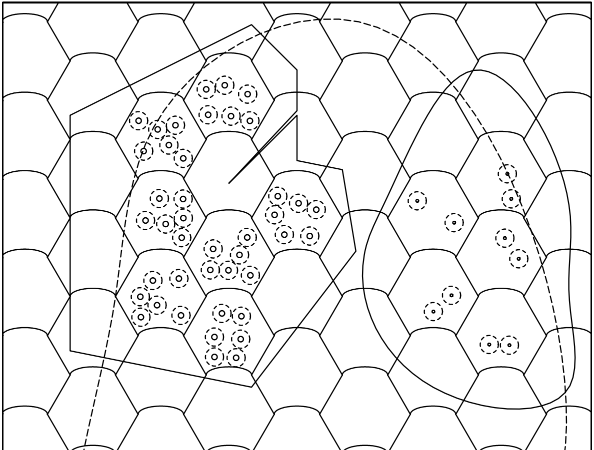
Result (I. arXiv:2208.04557)

$\mu = \frac{(d-2)S_d}{|A|} \sum_{k \leq m} \mu_k N_k 1_{F_k}$.

Proof (using Theorem 1)

$$B_{\varepsilon,k} := \bigsqcup_{i\in\Lambda_\varepsilon^-(F_k), j\le N_k} \overline{B(x_{i,j}^\varepsilon, C\varepsilon)}, \;\; B_\varepsilon := \bigsqcup_{k\le m} B_{\varepsilon,k}.$$

$$w^\varepsilon := \begin{cases} 0 & \text{on } T_\varepsilon \\ \text{harmonic} & \text{on } B_\varepsilon \setminus T_\varepsilon \\ 1 & \text{on } B_\varepsilon^c \end{cases} \in C(\mathbb{R}^d).$$



Approximation of sets

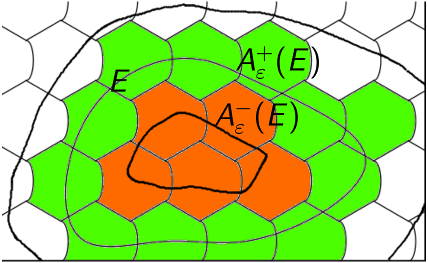
For $E \subset \mathbb{R}^d, \Lambda_\varepsilon^+(E) := \{i \in \Lambda \mid A_i^\varepsilon \cap E \neq \emptyset\}$.
$A_\varepsilon^\pm(E) := \bigsqcup_{i \in \Lambda_\varepsilon^\pm(E)} A_i^\varepsilon$.
(Recall $\Lambda_\varepsilon^-(E) = \{i \in \Lambda \mid A_i^\varepsilon \subset E\}$)

Lemma 3

$\forall E \in \mathcal{J}_b,\ |A_\varepsilon^\pm(E)| \to |E|. \# \Lambda_\varepsilon^\pm(E) = \frac{|A_\varepsilon^\pm(E)|}{\varepsilon^d |A|}$.

Proof.

$d_\varepsilon := \mathrm{diam} \varepsilon A$. $\{x \in \mathbb{R}^d \mid \mathrm{dist}(x, (\mathring{E})^c) > d_\varepsilon\} \subset A_\varepsilon^-(\mathring{E}) \subset E$
$\subset A_\varepsilon^+(E) \subset \overline{\bigcup_{x \in E} B(x, d_\varepsilon)}$. □



$w^\varepsilon \to 1$ weakly (1/3)

w^ε with polar coordinate is $w^\varepsilon_{0,k}(r) = \frac{(a_{\varepsilon,k})^{-d+2} - r^{-d+2}}{(a_{\varepsilon,k})^{-d+2} - (C\varepsilon)^{-d+2}}$.

$$\|\nabla w^\varepsilon\|^2_{L^2(A_i^\varepsilon)} = \begin{cases} \frac{N_k S_d (d-2)}{(a_{\varepsilon,k})^{-d+2} - (C\varepsilon)^{-d+2}} & (i \in \Lambda^-_\varepsilon(F_k)) \\ 0 & (i \notin \bigsqcup_k \Lambda^-_\varepsilon(F_k)) \end{cases}.$$

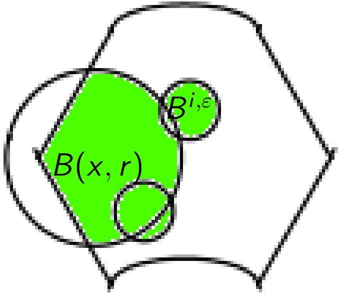
(radii) gives $c := \sup_{\varepsilon,i} \varepsilon^{-d} \|\nabla w^\varepsilon\|^2_{L^2(A_i^\varepsilon)} < \infty$.

$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leq \|\nabla w^\varepsilon\|^2_{L^2(A^+_\varepsilon(\Omega))} \leq \#\Lambda^+_\varepsilon(\Omega) c \varepsilon^d \leq \frac{c|A^+_\varepsilon(\Omega)|}{|A|}$.

$|w^\varepsilon| \leq 1$. $\therefore \sup_{\varepsilon<1} \|w^\varepsilon\|_{H^1_0(\Omega)} < \infty$.

$w^\varepsilon \to 1$ weakly (2/3)

For $i \notin \bigsqcup_k \Lambda_\varepsilon^-(F_k)$, $B^{i,\varepsilon} := \emptyset . N := \max_k N_k$.
$B^{i,\varepsilon} \subset \exists D_i^\varepsilon \subset A_i^\varepsilon$, $|D_i^\varepsilon| = N|B(0, C\varepsilon)|$.
(shown by intermediate theorem for
$r \mapsto |((A_i^\varepsilon \setminus B^{i,\varepsilon}) \cap B(x, r)) \sqcup B^{i,\varepsilon}|$)



$g_\varepsilon := 1_{(\bigcup_i D_i^\varepsilon)^c}$. $w^\varepsilon g_\varepsilon = g_\varepsilon$.

$w^{\varepsilon} \to 1$ weakly (3/3)

Lemma 4

$f \in L^{\infty}, \sup_{n\in\mathbb{N}} \|f_n\|_{\infty} < \infty, \forall E \in \mathcal{J}_b, \langle f_n - f, 1_E \rangle_{L_1^*} \to 0$
$\Rightarrow \forall h \in L^1, \langle f_n - f, h \rangle \to 0.$

We apply Lemma 4 for $w^{\varepsilon} g_{\varepsilon} = g_{\varepsilon}$.

$\left\langle g_{\varepsilon}, 1_{A_{\varepsilon}^{-}(E)} \right\rangle \leq \langle g_{\varepsilon}, 1_E \rangle \leq \left\langle g_{\varepsilon}, 1_{A_{\varepsilon}^{+}(E)} \right\rangle.$

$\left\langle g_{\varepsilon}, 1_{A_{\varepsilon}^{\pm}(E)} \right\rangle = (|\varepsilon A| - N|B(0, C\varepsilon)|)\#\Lambda_{\varepsilon}^{\pm}(E)$

$\to \langle c, 1_E \rangle$ where $c = \frac{|A|-N|B(0,C)|}{|A|}$.

For any subsequence w^{ε_n} converge weakly, and limit w,

$\langle g_{\varepsilon_n} w^{\varepsilon_n} - cw, 1_E \rangle = \int g_{\varepsilon_n}(w^{\varepsilon_n} - w) 1_E dx + \langle g_{\varepsilon_n} - c, w 1_E \rangle \to 0.$

$\therefore cw = c, \ w = 1.$

$\langle -\Delta w^\varepsilon, v^\varepsilon \rangle \to \langle \mu, v \rangle$ (1/2)

Property of $w^\varepsilon, v^\varepsilon$ gives$\langle -\Delta w^\varepsilon, v^\varepsilon \rangle$
$= \sum_{k \le m} \frac{(w^\varepsilon_{0,k})'(C\varepsilon)}{C\varepsilon} \left(\int_{B_{\varepsilon,k}} \nabla q^\varepsilon \cdot \nabla v^\varepsilon dx + d \left\langle 1_{B_{\varepsilon,k}}, v^\varepsilon \right\rangle \right)$
where $\Delta q^\varepsilon = d$ on $B_\varepsilon, q^\varepsilon = 0$ on $B^c_\varepsilon . \|\nabla q^\varepsilon\|_\infty \le C\varepsilon$.
$\frac{(w^\varepsilon_{0,k})'(C\varepsilon)}{C\varepsilon} \to \frac{(d-2)\tilde{\mu_k}}{C^d}$.
$\left| \int_{B_{\varepsilon,k}} \nabla q^\varepsilon \cdot \nabla v^\varepsilon dx \right| \le C\varepsilon \|v^\varepsilon\|_{W^{1,1}(\Omega)} \to 0.$

$\langle -\Delta w^\varepsilon, v^\varepsilon \rangle \to \langle \mu, v \rangle$ (2/2)

$\forall E \in \mathcal{J}_b, B_{\varepsilon,k} \subset F_k$ gives

$\left\langle 1_{B_{\varepsilon,k}}, 1_{A_\varepsilon^-(E\cap F_k)} \right\rangle \leq \left\langle 1_{B_{\varepsilon,k}}, 1_E \right\rangle \leq \left\langle 1_{B_{\varepsilon,k}}, 1_{A_\varepsilon^+(E\cap F_k)} \right\rangle.$

$\left\langle 1_{B_{\varepsilon,k}}, 1_{A_\varepsilon^\pm(E\cap F_k)} \right\rangle = N_k |B(0, C\varepsilon)| \# \Lambda_\varepsilon^\pm (E \cap F_k) \to$

$\frac{N_k|B(0,C)||F_k \cap E|}{|A|} = \left\langle \frac{N_k|B(0,C)|}{|A|} 1_{F_k}, 1_E \right\rangle.$

$\left\langle 1_{B_{\varepsilon,k}}, v^\varepsilon \right\rangle = \left\langle 1_{B_{\varepsilon,k}}, v^\varepsilon - v \right\rangle + \left\langle 1_{B_{\varepsilon,k}}, v \right\rangle$

$\overset{Lemma\ 4}{\longrightarrow} \left\langle \frac{N_k|B(0,C)|}{|A|} 1_{F_k}, v \right\rangle.$

$\therefore \langle -\Delta w^\varepsilon, v^\varepsilon \rangle \to \left\langle \frac{(d-2)S_d}{|A|} \sum_{k \leq m} \mu_k N_k 1_{F_k}, v \right\rangle.$

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