Random Schrödinger operators with complex decaying potentials

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Himeji Conference on PDEs 2023
March 04, 2023

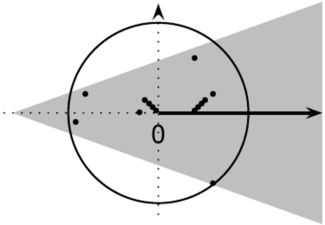
Schrödinger operators with complex potentials

Let $d \in \mathbb{N}$ and consider

$$-\Delta + V \quad \text{in } L^2(\mathbb{R}^d) \quad \text{with } V \in L^q(\mathbb{R}^d : \mathbb{C}) \quad \text{and} \quad \begin{cases} d = 1\,, & q \in [1, \infty)\,, \\ d = 2\,, & q \in (1, \infty)\,, \\ d \geq 3\,, & q \in [d/2, \infty) \end{cases}$$

realized as m-sectorial operator (Friedrichs).

We study $\sigma(-\Delta + V) \cap (\mathbb{C} \setminus [0, \infty))$, which (since $|V|^{\frac{1}{2}}(-\Delta + 1)^{-\frac{1}{2}} \in \mathcal{S}^\infty$) consists of discrete eigenvalues of finite algebraic multiplicities which can only accumulate at $[0, \infty)$.



Estimates for individual eigenvalues

For real-valued V the Keller-Lieb–Thirring ('61, '76) inequality

$$\sum_{\lambda \in \sigma(-\Delta+V)\setminus[0,\infty)} |\lambda|^{q-d/2} \lesssim_{d,q} \int_{\mathbb{R}^d} |V|^q \quad \text{for} \quad \begin{cases} d=1\,, & q \geq 1\,, \\ d=2\,, & q>1\,, \\ d \geq 3\,, & q \geq d/2 \end{cases}$$

sheds light on the lowest eigenvalue and the accumulation of negative eigenvalues at 0.

Corresponding inequality for complex V?

Let $z \in \sigma_d(-\Delta + V)$. In $d = 1$, Abramov–Aslanyan–Davies ('01) showed

$$|z|^{\frac{1}{2}} \lesssim \int_{\mathbb{R}} |V|.$$

For $d \geq 2$ and $q \geq \max\{1+, \frac{d}{2}\}$, Frank–Laptev–Lieb–Seiringer ('06) got

$$|z|^{q-\frac{d}{2}} \lesssim_{d,q} \left(1 + \frac{\mathrm{Re}(z)_+}{|\mathrm{Im}(z)|}\right)^q \int_{\mathbb{R}^d} |V|^q.$$

Conjecture (Laptev–Safronov ('09))

Any eigenvalue $z \in \mathbb{C}\setminus[0,\infty)$ of $-\Delta + V$ satisfies

$$|z|^{q-\frac{d}{2}} \lesssim_{d,q} \int_{\mathbb{R}^d} |V|^q , \qquad d \geq 2 , \quad q \in (d/2, d] . \qquad \text{(LT)}$$

Status:

- Bögli ('17) constructed a radial $V \in L^q$ with $q > d$ and $\|V\|_q < \varepsilon$ such that $(1 + i\varepsilon) \in \sigma(-\Delta + V) \setminus [0, \infty)$.
  $\Rightarrow$ *(LT) false for $q > d$.*
- Frank ('11) proved (LT) for all $q \leq (d+1)/2$ via Kenig–Ruiz–Sogge ('83).
- Bögli–Cuenin ('21): *(LT) false for $q > (d+1)/2$.* (Inspired by Fourier restriction.)

How "special" is the BC example?

Random potentials

Consider eigenvalues of realizations of $-\Delta + V_\omega$ in $L^2(\mathbb{R}^d)$ with

$$V_\omega(x) = \sum_{j \in h\mathbb{Z}^d} \omega(j) V(x) \mathbb{1}_{[0,1)^d}\left(\frac{x-j}{h}\right),$$

where

- $\omega(j)$ are iid. symmetric Bernoulli or Gaussian random variables,
- $h > 0$ is a randomization scale.

Main result

Theorem 1 (Cuenin–M., arXiv:2201.04466)

For any $q < d+1$, there exist constants $M_0, c > 0$ s.t. the following holds.
For any $h, \lambda > 0$, $|\varepsilon| \ll \lambda$, for any $V \in L^q(\mathbb{R}^d)$, and for any $M \geq M_0$, each eigenvalue $z = (\lambda + i\varepsilon)^2$ of $-\Delta + V_\omega$ satisfies

$$\frac{\lambda^{2-\frac{d}{q}}}{\langle \lambda h\rangle^{d/2}(\ln\langle \lambda h\rangle)^2} \lesssim_{d,q} M\|V\|_{L^q(\mathbb{R}^d)}$$

except for ω in a set of measure $< \exp(-cM^2)$.

Remark: The BC example V_ε obeys $|V_\varepsilon| \lesssim \varepsilon \mathbb{1}_{T_\varepsilon}$ with $T_\varepsilon = \{(x_1, x') : |x_1| < \varepsilon^{-1}, |x'| < \varepsilon^{-\frac{1}{2}}\}$, $\|V_\varepsilon\|_q \lesssim \varepsilon^{1-\frac{d+1}{2q}}$, and generates an eigenvalue $1 + i\varepsilon$ of $-\Delta + V_\varepsilon$.
$\Rightarrow$ BC example is almost surely destroyed after randomizing on the scale $h < \varepsilon^{(\frac{d+1}{2q}-1)\cdot\frac{2}{d}}$.

Because $1 = \lambda^{2-d/q} = \frac{\lambda^{2-\frac{d}{q}}}{\langle\lambda h\rangle^{d/2}(\ln\langle\lambda h\rangle)^2} \cdot \langle\lambda h\rangle^{d/2}(\ln\langle\lambda h\rangle)^2 \lesssim \|V\|_q\langle\lambda h\rangle^{d/2}(\ln\langle\lambda h\rangle)^2$ and the right side is $\gtrsim 1$ for $h < \varepsilon^{(\frac{d+1}{2q}-1)\cdot\frac{2}{d}}$.

Spectral radius

Recall

$$z \in \sigma(-\Delta + V_\omega) \Rightarrow 1 \le \mathrm{spr}(R(z)^{1/2} V_\omega |R(z)|^{1/2}) = \lim_{n\to\infty} \|(R(z)^{\frac{1}{2}} V_\omega |R(z)|^{\frac{1}{2}})^n\|^{1/n}$$

with $R(z) = (-\Delta - z)^{-1}$. Wlog, $z = (1 + i\varepsilon)^2$.

For simplicity, assume $V \in \langle x\rangle^{-\iota} L^q$ with $q \le d + 1$. Decompose

$$V_\omega(x) = V_0(x) + \sum_{\ell\ge 1} V_\ell(x)\,, \quad \text{where}$$

$$V_0(x) = V_\omega(x)\mathbb{1}_{\{|x|\le 1\}}\,, \quad V_\ell(x) = V_\omega(x)\mathbb{1}_{\{2^{\ell-1}\le|x|\le 2^\ell\}}\,.$$

Consider

$$\sum_{\ell_1,\dots,\ell_n=0}^{\infty} R(z)^{\frac{1}{2}} V_{\ell_1} R(z)^{\frac{2}{2}} V_{\ell_2} R(z)^{\frac{2}{2}} \cdots V_{\ell_{n-1}} R(z)^{\frac{2}{2}} V_{\ell_n} |R(z)|^{\frac{1}{2}}\,.$$

Elementary operators

Position cut off $\Rightarrow$ frequency smoothing on inverse scale

$$\mathbb{1}_{\{|x|\le 2^{\ell_j}\}} R(z) \mathbb{1}_{\{|x|\le 2^{\ell_{j+1}}\}} = \mathbb{1}_{\{|x|\le 2^{\ell_j}\}} \mathcal{F}^{-1} \left(\frac{1}{|\xi|^2 - z} * \gamma_{\delta_j} \right) \mathcal{F} \mathbb{1}_{\{|x|\le 2^{\ell_{j+1}}\}}$$

where $\delta_j^{-1} > 2^{\ell_j} + 2^{\ell_{j+1}}$ and $\gamma_{\delta_j}(\xi) = \delta_j^{-d}\gamma(\xi/\delta_j)$ is Schwartz with $\gamma^\vee \in C_c^\infty$.

Frequencies $||\xi|^2 - 1| \gg 1$ are harmless (Sobolev).

By the coarea formula for $|\xi|^2 \sim 1$ and Cauchy–Schwarz, it suffices to estimate

$$\ln^{\frac{1}{2}} \frac{1}{\delta_j} \cdot \ln^{\frac{1}{2}} \frac{1}{\delta_{j+1}} \cdot \boxed{\sup_{t,t' \in (1/2,2)} \| F_{M_t} V_{\ell_j} F^*_{M_{t'}} \|_{L^2(M_{t'}),L^2(M_t)}}$$

where $M_t := \{\xi \in \mathbb{R}^d : |\xi| = t\}$ with associated Fourier restriction and extension operators

$$(F_M \psi)(\xi) := \int_{\mathbb{R}^d} dx \, \mathrm{e}^{-2\pi i x\cdot\xi} \psi(x)\big|_{\xi \in M}, \quad (F^*_M \phi)(x) := \int_M d\sigma_M(\xi) \, \mathrm{e}^{2\pi i x\cdot\xi} \phi(\xi).$$

Local extension bound

Stein–Tomas: $\|F_{M_t} V F^*_{M_{t'}}\| \lesssim \|V\|_{\frac{d+1}{2}}$. The exponent $\frac{d+1}{2}$ is optimal.

Randomness allows to halve the decay!

Theorem 2

Let $q \le d+1$, $R \ge h$, *and* $V_\omega(x) = \sum_{j \in h\mathbb{Z}^d} \omega(j)\, V(x) \mathbb{1}_{[0,1)^d}\left(\frac{x-j}{h}\right)$ *with* $\operatorname{supp} V_\omega \subseteq B(R)$. *Then*

$$\mathbb{E}\|F_{M_t} V_\omega F^*_{M_{t'}}\| \lesssim \langle h\rangle^{\frac{d}{2}} (\ln\langle R\rangle)^{\frac{1}{2}} (\ln\langle h\rangle + \ln\langle R\rangle)^2 \|V\|_q, \quad t, t' \in (1/2, 2).$$

By the tail bound $\mathbb{P}(\|X\| > t) \le \exp(-\frac{ct^2}{(\mathbb{E}\|X\|)^2})$, this shows that

$$\ln^{\frac{1}{2}} \frac{1}{\delta_j} \cdot \ln^{\frac{1}{2}} \frac{1}{\delta_{j+1}} \cdot \|F_{M_t} V_{\ell_j} F^*_{M_{t'}}\|$$

$$\lesssim M(\ell_{j-1} + \ell_j + \ell_{j+1}) \cdot (\ell_{j-1} + \ell_j + \ell_{j+1}) \cdot \langle h\rangle^{\frac{d}{2}} \ell_j^{1/2} (\ln\langle h\rangle + \ell_j)^2 \cdot 2^{-\iota \ell_j} \|\langle x\rangle^{\iota} V\|_q$$

holds for all $\{\ell_j\}_j$ and ω outside a set of measure

$$< \sum_{\ell_{j-1}, \ell_j, \ell_{j+1}} \exp\left(-cM^2(\ell_{j-1} + \ell_j + \ell_{j+1})^2\right) = \mathrm{e}^{-cM^2}.$$

Rough ideas to get the doubling of the exponent $(d+1)/2 \mapsto d+1$

- Exploit square root cancellation via Dudley's inequality (cf. Khintchine)

$$\mathbb{E} \sup_{(a(n))_n \subset \mathcal{A}} \left| \sum_n \omega(n) a(n) \right| \lesssim \sqrt{\ln(|\mathcal{A}|)} \sup_{(a_n)_n \subset \mathcal{A}} \left(\sum_n |a(n)|^2 \right)^{1/2} .$$

- *Problem:* we are dealing with $\sup_{g,g' \in L^2(M)} |\langle g, F_M V_\omega F_M^* g' \rangle_{L^2(B(R))}|$
- *However:* we are frequency localized on unit scale $\Rightarrow$ discretize x-space on unit scale
- *Moreover:* we are position localized on scale R $\Rightarrow$ discretize ξ-space on scale R^{-1}
- The discretization allows us to reduce the analysis to

$$\mathbb{E} \sup_{\|g\|_{\ell^2(\Lambda_R^*)}, \|g'\|_{\ell^2(\Lambda_R^*)} \leq R^{-(d-1)/2}} |\langle F_{M,d}^* g, v_\omega F_{M,d}^* g' \rangle_{\ell^2(B(R) \cap \mathbb{Z}^d)}|$$

  where $F_{M,d}^* : \ell^2(\Lambda_R^*) \to \ell^\infty(B(R) \cap \mathbb{Z}^d)$ is a discretized Fourier extension operator on a R^{-1}-net $\Lambda_R^* \subseteq M$ in ξ-space and $v_\omega(n) = \omega(n) v(n)$ with $v \in \ell^q(B(R) \cap \mathbb{Z}^d)$.

  Now: replace sup over infinite set by a sup over a finite set by paying an entropy cost.

- To treat the supremum over $\ell^2(\Lambda_R^*)$, we use Bourgain's ('02) idea: **covering & chaining/telescoping:**
  $\exists \mathcal{F}_k \subseteq \mathrm{ran}(F^*_{M,d}) \subseteq \ell^\infty(B(R)\cap\mathbb{Z}^d)$ s.t. for all $g \in \ell^2(\Lambda_R^*)$ with $\|g\|_{\ell^2(\Lambda_R^*)} \leq R^{-\frac{d-1}{2}}$,

  $$F^*_{M,d} g = \sum_{k\geq 0} \xi^{(k)} \quad \text{for some } \xi^{(k)} \in \mathcal{F}_k$$

  with $\|\xi^{(k)}\|_\infty \leq 2^{-k} R^{-\frac{d-1}{2}}$ and $\|\xi^{(k)}\|_{p'} \lesssim R^{-\frac{d-1}{2}}$ with $p' \geq 2(d+1)/(d-1)$.
- It suffices to estimate

  $$\sum_{k,k'\geq 0} \mathbb{E} \sup_{\mathcal{F}_k \times \mathcal{F}_{k'}} |\langle \xi^{(k)}, v_\omega \xi^{(k')}\rangle_{\ell^2(B(R)\cap\mathbb{Z}^d)}|$$

- Thanks to the dual Sudakov entropy bound

  $$\boxed{\ln|\mathcal{F}_k| \lesssim 4^k \|F^*_{M,d}\|^2_{\ell^2(\Lambda_R^*)\to\ell^\infty(B(R)\cap\mathbb{Z}^d)} \ln(|B(R)\cap\mathbb{Z}^d|)}$$

  (Pajor–Tomczak-Jaegermann ('86)), Dudley's inequality yields

  $$\mathbb{E} \sup_{\mathcal{F}_k \times \mathcal{F}_{k'}} |\langle \xi^{(k)}, v_\omega \xi^{(k')}\rangle_{\ell^2(B(R)\cap\mathbb{Z}^d)}| \lesssim \sqrt{\ln(R)}\|v\|_q, \quad q \leq d+1.$$

- Interpolating the random bound with the deterministic bound (Hölder)

$$\sup_{\mathcal{F}_k \times \mathcal{F}_{k'}} |\langle \xi^{(k)}, v_\omega \xi^{(k')} \rangle_{\ell^2(B(R)\cap\mathbb{Z}^d)}| \lesssim R^{d-d/q} 2^{-k-k'} \|v\|_q, \quad q \geq 1$$

allows to conclude the proof of Theorem 2.

Local to global arguments – Pf. of Thm. 1 ($h = 1$)

To remove the $\langle x\rangle^{-\iota}$ decay assumption, use sparse decomposition of V.

- **Horizontal dyadic decomposition** of V:

$$V = \sum_{i\geq 0} V_i\,,$$

where $V_i = V\mathbb{1}_{H_{i+1}<|V|<H_i}$ (with $H_i = \inf\{t>0 : |\{|V|>t\}| \leq 2^{i-1}\}$).
Note $|\mathrm{supp}(V_i)| = 2^i$ and $\|V\|_{L^{q,r}} \sim \|H_i 2^{i/q}\|_{\ell^r_i(\mathbb{N})}$.

- **Sparse CZ decomposition** of each V_i (Tao ('99)[1]): Let

$$K_i = \mathcal{O}(K2^{i/K})\,,\quad N_i = \mathcal{O}(2^i)\,,\quad R_i = \mathcal{O}(2^{i\gamma^K})$$

with $K \gg 1$ and $\gamma > 0$ arbitrary but fixed.

Then $\mathrm{supp}(V_i)$ is covered by K_i many **sparse collections of balls**, each containing at most N_i many balls. **Sparse** means that the centers of the balls $\{B(x_k, R_i)\}_{k=1}^{N_i}$ in the same collection are $(R_i N_i)^\gamma$-separated. In particular,

$$V = \sum_{i\geq 0}\sum_{j=1}^{K_i}\sum_{k=1}^{N_i} V_{ijk}\,.$$

[1] See also Pinney ('21) for a pedagogical introduction.

Probabilistic bound: in the multilinear expansion of the spectral radius, we obtain (similarly as before)

$$\begin{aligned}&\|C^{(\delta_1)}\mathbb{1}_{B_2}V_\omega C^{(\delta_2)}\| \\ &\quad\lesssim M\,[\log(1/\delta_1+1/\delta_2)]^{\mathcal{O}(1)}\cdot[\log(1/\delta_1+1/\delta_2)]^{\mathcal{O}(1)}\,\|V\mathbf{1}_{B_2}\|_{L^q}\end{aligned}$$

for all ω outside a set of measure at most

$$\sum_{i_1,i_2,i_3} N_{i_1}K_{i_1}N_{i_2}K_{i_2}N_{i_3}K_{i_3}\exp(-c'M^2\,[\log(1/\delta_1+1/\delta_2)]^{\mathcal{O}(1)})\lesssim \mathrm{e}^{-cM^2}$$

Here $B_k=B(x_k,R_k)$ are arbitrary balls and $C^{(\delta)}(D)$ obeys

$$|C^{(\delta)}(\xi)|\lesssim(||\xi|^2-1|+\delta)^{-1/2}$$

with

$$\delta_1=\langle d(B_1,B_2)+2R_1+2R_2\rangle^{-1},\quad \delta_2=\langle d(B_2,B_3)+2R_2+2R_3\rangle^{-1}.$$

Writing $\alpha_\ell=(i_\ell,j_\ell,k_\ell)$ for $\ell\in\mathbb{N}_0$, we get

$$\|R_0V_{\alpha_1}R_0V_{\alpha_2}\dots R_0V_{\alpha_n}\|\lesssim AM^n\prod_{\ell=1}^{n}[\log(1/\delta_{\alpha_\ell})+\log(1/\delta_{\alpha_{\ell+1}})]^{C}\|V_{\alpha_\ell}\|_q$$

for all $q\le d+1$ except for ω in a set of measure at most $\exp(-cM^2)$.

Deterministic bound: exploit

$$|(-\Delta - \mathrm{e}^{i\phi})^{-a+it}(x,y)| \lesssim \mathrm{e}^{ct^2}|x-y|^{-\frac{d-1}{2}+a}$$

and use complex interpolation to obtain, with $q_\eta = \frac{d+1}{2} - \eta$ and $\eta' = \frac{\eta}{q_\eta}$,

$$\|V_\alpha^{\frac{1}{2}} R_0 |V_\beta|^{\frac{1}{2}}\| \lesssim (\delta_{\alpha\beta} + d(B_\alpha, B_\beta))^{-\eta'} \|V_\alpha\|_{q_\eta}^{1/2} \|V_\beta\|_{q_\eta}^{1/2}.$$

Borrow an ε of this and **interpolate** ($\theta \in (0,1)$) with random bound
$\Rightarrow$ with high probability:

$$\sum_{j_1,\dots,j_n} \sum_{k_1,\dots,k_n} \|R_0 V_{\alpha_1} R_0 V_{\alpha_2} \dots R_0 V_{\alpha_n}\|$$

$$\lesssim M^n \prod_{\ell=1}^{n} [\log(1 + R_{i_{\ell-1}} + R_{i_\ell} + R_{i_{\ell+1}})]^{\mathcal{O}(1)} \times K_{i_\ell} \times H_{i_\ell} 2^{i_\ell((1-\theta)/q + \theta/q_\eta)}.$$

We used $(\delta_{\alpha_\ell,\alpha_{\ell+1}} + d(B_{\alpha_\ell}, B_{\alpha_{\ell+1}}))^{-\frac{\theta\eta'}{2}}$ to control $[\log(d(B_{\alpha_\ell}, B_{\alpha_{\ell+1}}))]^{\mathcal{O}(1)}$ in $\log(1/\delta_{\alpha_\ell})$.
Moreover, we used $\sum_{k_1 \le N_{i_1}} \langle d(B(x_{k_1}, R_{i_1}), B_{\alpha_2})\rangle^{-\theta\eta'/2} \lesssim_\gamma 1$ uniformly in i_1, j_1, i_2, j_2, k_2, provided $\theta\eta'\gamma/2 > 1$ (so take $\gamma \gg 1$ depending on the final values of η, θ) thanks to $d(B(x_{k_1}, R_{i_1}), B_{\alpha_2}) \ge \frac{1}{2}(N_{i_1} R_{i_1})^\gamma$ for all but at most one k_1.
(If this did not hold for two distinct k_1, k_1', then by the triangle inequality, $d(B(x_{k_1}, R_{i_1}), B(x_{k_1'}, R_{i_1})) < (N_{i_1} R_{i_1})^\gamma$, which contradicts the sparsity of the collection $\{B(x_{k_1}, R_{i_1})\}$.)

Recall $q < d+1$, $q_\eta = \frac{d+1}{2} - \eta$, and $\theta\eta'\gamma/2 > 1$. Once K is fixed, choose η, θ such that $0 < \theta(1/q_\eta - 1/q) < 1/K$. Since $K_i[\log(1+R_{i_\ell})]^{\mathcal{O}(1)} \lesssim 2^{2i/K}$, we obtain

$$\mathrm{spr}(BS(z)) \lesssim \sum_{i\in\mathbb{Z}_+} H_i 2^{i/q} 2^{3i/K}\,, \quad q < d+1\,.$$

Use this for $\tilde{q} > q$ instead of q, i.e., we now regard $(d+1)/2 < q < d+1$ as given and choose $\tilde{q} < d+1$ and K such that $1/\tilde{q} + 3/K < 1/q$. Then

$$\mathrm{spr}(BS(z)) \lesssim_q \sup_{i\in\mathbb{Z}_+} H_i 2^{i/q} \sum_{i\in\mathbb{Z}_+} 2^{i(1/\tilde{q}-1/q+3/K)} \lesssim \|V\|_{L^{q,\infty}}\,.$$

Remark: In a similar vein, we obtain

$$\|F_{M_t} V_\omega F^*_{M_{t'}}\| \lesssim M\langle h\rangle^{d/2}(\log\langle h\rangle)^2 \|V\|_{L^q}\,, \quad q < d+1$$

except for ω in a set of measure at most $\exp(-cM^2)$.

Outlook

- What happens when randomness is lacunary?
- Is the Laptev–Safronov conjecture true **generically** *(in a non-random sense)?*
  Aim: find suitable generic condition for the potential that prevents Knapp examples at many scales.
- Optimality of $V \in L^{d+1-\varepsilon}$ in view of better restriction estimates (as in $d = 2$) and considering instead $R(z)^{1/2} V R(z) V |R(z)|^{1/2}$ as basic block?
- Upgrade operator norm estimates for $F_M V_\omega F_M^*$ to Schatten class estimates (with high probability)? (cf. Safronov ('21))

THANK YOU FOR LISTENING!

Proof of Theorem 2

Ingredients in proof of Theorem 2

- Our potential is chopped up into compactly supported pieces. Let R be the length scale of one such (dyadic) chunk.

- $\Rightarrow$ blurring in ξ-space on the inverse scale R^{-1} and "locally constant" properties
  $\Rightarrow$ discretization of ξ-space on the scale R^{-1}, discrete restriction theory available

- We are also localized in ξ-space on the unit scale $\mu = 1$
  $\Rightarrow$ locally constant property and discretization in x-space on inverse scale $\mu^{-1} = 1$.

- $\mathbb{E}\sup_{g,g' \in L^2(M)} |(g, F_M V_\omega F_M^* g')|$: supremum is over infinite-dimensional set.
  Discretization of x-space allows to reduce this to computing supremum over a finite set,

$$\mathbb{E} \sup_{(a(j))_j \in \mathcal{A}} |\sum_j \omega(j) a(j)| \lesssim \sqrt{\ln(|\mathcal{A}|)} \sup_{(a(j))_j \in \mathcal{A}} (\sum_j |a(j)|^2)^{1/2}.$$

by Dudley's inequality. An entropy bound (dual Sudakov) allows to control $\ln(|\mathcal{A}|)$.

Proof of Theorem 2

In the proof of Theorem 2 we substitute $q \mapsto 2q$ for convenience. That is, throughout the proof, we assume $q \leq (d+1)/2$.
We compute

$$\mathbb{E} \sup_{g,g'} |(g, F_{M_t} V_\omega F^*_{M_{t'}} g')| = \mathbb{E} \sup_{g,g'} \left| \int_{\mathbb{R}^d} \overline{(F^*_{M_t} g)(x)} V_\omega(x) (F^*_{M_{t'}} g')(x) \right|$$

for $g \in L^2(M_t)$ and $g' \in L^2(M_{t'})$.

- Let $\Lambda^*_R := \{\eta_\nu\}_\nu \subseteq M_t$ be a $1/R$-net in ξ-space. Write $\xi = \eta_\nu + \tau$ with $\tau \in M_t \cap B_0(\frac{1}{R})$.
- Let $\Lambda_\mu = \{x_i\}_i := \mu^{-1}\mathbb{Z}^d$ be a $1/\mu$-net in x-space. Write $x = x_i + y$ with $y \in Q_{1/\mu}$.

By partition of unity we can assume that g is supported on a disjoint union of balls $B(\eta_\nu, \frac{1}{R})$. Then

$$(F^*_{M_t} g)(x) = \int_{M_t \cap B_0(\frac{1}{R})} d\sigma(\tau) \sum_\nu \exp(2\pi i (x_i + y) \cdot (\eta_\nu + \tau)) g(\eta_\nu + \tau)$$

Similarly, by a partition of unity we have for any $G \in L^1(\mathbb{R}^d : \mathbb{C})$,

$$\int_{\mathbb{R}^d} G(x)\, dx = \sum_i \int_{Q_{1/\mu}} G(x_i + y)\,.$$

Let $\|f\|_{\ell^2_{sc}(\Lambda_R^*)} := R^{\frac{d-1}{2}} \|f\|_{\ell^2(\Lambda_R^*)}$ and introduce the **discrete extension operator**

$$\begin{aligned} S_t : \ell^2_{sc}(\Lambda_R^*) &\to \ell^\infty(\Lambda_\mu \cap B_R) \\ \{g(\eta_\nu + \tau)\}_\nu &\mapsto \{\sum_{\eta_\nu \in \Lambda_R^*} e((x_i + y)(\eta_\nu + \tau))g(\eta_\nu + \tau)\}_i \,. \end{aligned}$$

By discrete restriction theory (with $t \in (1/2, 2)$ and $1/q = 1/p - 1/p'$)

$$\|S_t\|_{\ell^2_{sc}(\Lambda_R^*) \to \ell^{p'}(\Lambda_\mu \cap B_R)} \lesssim t^{(-1+d/q)/2} \mu^{\frac{d}{p'}} \sim \mu^{\frac{d}{p'}}$$
$$\|S_t\|_{\ell^2_{sc}(\Lambda_R^*) \to \ell^\infty(\Lambda_\mu \cap B_R)} \lesssim t^{(-1+d/1)/2} \mu^{\frac{d}{\infty}} \sim 1$$

The discretization in x-space and the $\ell^2_{sc}(\Lambda_R^*) \to \ell^\infty(\Lambda_\mu \cap B_R)$-boundedness of S allow us to reduce the computation of the supremum over an infinite-dimensional set to the computation of the supremum over a finite set, whose cardinality we can control.

There are sets $\mathcal{F}_k \subseteq \ell^\infty(\Lambda_\mu \cap B_R)$ such that we can represent

$$\sum_\nu e((x_i + y)(\eta_\nu + \tau))g(\eta_\nu + \tau) \equiv \sum_{k\geq 0} \xi_i^{(k)}, \quad \xi_i^{(k)} \in \mathcal{F}_k \subseteq \ell^\infty(\Lambda_\mu \cap B_R)$$

(the $\xi_i^{(k)}$ also depend on y, τ) with $\ln|\mathcal{F}_k| \lesssim \ln(\mu R)4^k t^{d-1} \sim \ln(\mu R)4^k$ and

$$\|\xi_i^{(k)}\|_{\ell_i^{p'}(\Lambda_\mu \cap B_R)} \lesssim t^{(-1+d/q)/2}\mu^{d/p'}\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{sc}}} \sim \mu^{d/p'}\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{sc}}}$$

$$\|\xi_i^{(k)}\|_{\ell_i^{\infty}(\Lambda_\mu \cap B_R)} \lesssim 2^{-k} \cdot t^{(-1+d/1)/2}\mu^{d/\infty}\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{sc}}} \sim 2^{-k}\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{sc}}}$$

Thus,

$$\mathbb{E}\sup_{g,g'} |(g, F_{M_t} V_\omega F^*_{M_{t'}} g')|$$

$$\lesssim \sum_{k,k'\geq 0} \int_{Q_{1/\mu}} dy \int_{M_t \cap B_0(\frac{1}{R})} d\tau \int_{M_{t'} \cap B_0(\frac{1}{R})} d\tau' \, \mathbb{E} \max_{\mathcal{F}_k \times \mathcal{F}_{k'}} \left| \sum_{j\in h\mathbb{Z}^d} \omega(j) \sum_i \mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)} V(x_i+y)\xi_i^{(k)}\xi_i^{(k')} \right|$$

Lemmas

Let $q \leq (d+1)/2$ and
$X_{\xi,\xi'} := \left|\sum_{j\in h\mathbb{Z}^d} \omega(j) \sum_i \mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)} V(x_i+y)\xi_i^{(k)}\xi_i^{(k')}\right|.$

Lemma 3

$$\int_{B_0(\frac{10}{\mu})} dy \int_{M_t\cap B_0(\frac{10}{R})} d\tau \int_{M_{t'}\cap B_0(\frac{10}{R})} d\tau' \, \mathbb{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| \lesssim \sqrt{\ln(\mu R)} \cdot h^{\frac{d}{2}} \|V\|_{2q}$$

$$\int_{B_0(\frac{10}{\mu})} dy \int_{M_t\cap B_0(\frac{10}{R})} d\tau \int_{M_{t'}\cap B_0(\frac{10}{R})} d\tau' \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| \lesssim R^{d-\frac{d}{2q}} 2^{-(k+k')} \|V\|_{2q}$$

Lemma 4

Let $A > 0$. Then

$$\sum_{k,k'\geq 0} \min\{2^{-k-k'}, A\} \lesssim \begin{cases} A(1+(\ln_2(A))^2), & A \leq 1 \\ 1, & A \geq 1 \end{cases} .$$

The lemmas imply

$$\mathbb{E} \sup_{g \in L^2(M_t), g' \in L^2(M_{t'})} |\langle g, F_{M_t} V_\omega F^*_{M_{t'}} g' \rangle|$$

$$\lesssim \|V\|_{2q} R^{d - \frac{d}{2q}} \sum_{k,k' \geq 0} \min\{2^{-k-k'}, \sqrt{\ln(\mu R)} \cdot h^{\frac{d}{2}} R^{-d+\frac{d}{2q}}\}$$

$$\lesssim \|V\|_{2q} \langle h \rangle^{\frac{d}{2}} \sqrt{\ln(\mu R)} \left(1 + \ln^2 \left(\sqrt{\ln(\mu R)} \cdot h^{\frac{d}{2}} R^{-d+\frac{d}{2q}}\right)\right)$$

for all $t, t' \in (1/2, 2)$ and $\ell \geq 1$. This concludes the proof of Theorem 2.

Proof of Lemma 3 - Probabilistic estimate I

Recall $X_{\xi,\xi'} := \left|\sum_{j\in h\mathbb{Z}^d} \omega(j) \sum_i \mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)} V(x_i+y)\xi_i^{(k)}\xi_i^{(k')}\right|$.
By dual-to-Sudakov (later), $\ln(N) := \ln|\mathcal{F}_k \times \mathcal{F}_{k'}| \lesssim \ln(\mu R)(4^k + 4^{k'})$.
By Dudley,

$$\mathbb{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| \lesssim \sqrt{\ln N}\left(\sum_{j\in h\mathbb{Z}^d} \|\omega(j)\|_{\psi_2}^2 \Big|\sum_i V(x_i+y)\xi_i^{(k)}\xi_i^{(k')}\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\Big|^2\right)^{\frac{1}{2}}$$

$$\lesssim \sqrt{\ln N}\Big\|\|V(x_i+y)\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\|_{\ell_i^q}\Big\|_{\ell_j^{2q}} \cdot \Big\|\|\xi_i\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\|_{\ell_i^{p'}} \cdot \|\xi_i'\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\|_{\ell_i^{p'}}\Big\|_{\ell_j^{p'}}$$

with

$$\begin{aligned}
&\Big\|\|\xi\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\|_{\ell_i^{p'}} \cdot \|\xi'\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\|_{\ell_i^{p'}}\Big\|_{\ell_j^{p'}} \\
&\quad\le \min\Big\{\Big\|\|\xi_i\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\|_{\ell_i^{p'}}\Big\|_{\ell_j^\infty} \cdot \Big\|\|\xi_i'\mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)}\|_{\ell_i^{p'}}\Big\|_{\ell_j^{p'}}, \leftrightarrow\Big\} \\
&\quad\lesssim \mu^{\frac{d}{p'}}(\mu h)^{\frac{d}{p'}} \cdot \min\{2^{-k}, 2^{-k'}\}\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{sc}}(\Lambda_R^*)}\|g'(\eta_{\nu'}+\tau')\|_{\ell^2_{\nu',\mathrm{sc}}(\Lambda_R^*)}
\end{aligned}$$

Proof of Lemma 3 - Probabilistic estimate II

where we used $|\{i : x_i + y \in j + Q_h\}| < (\mu h)^d$ in the estimate of the ℓ_j^∞-norm and $|\{j \in h\mathbb{Z}^d : x_i + y \in j + Q_h\}| = 1$ to simplify the j-summation in the $\ell_j^{p'}$ norm.
By Hölder and $|\{i : x_i + y \in j + Q_h\}| < (\mu h)^d$ (recall $\{x_i\}_i = \Lambda_\mu$ is the $\frac{1}{\mu}$-net in B_R),

$$\begin{aligned}\int_{B_0(\frac{10}{\mu})} dy \, \big\| \| V(x_i + y) \mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)} \|_{\ell_i^q} \big\|_{\ell_j^{2q}} &\leq \int_{B_0(\frac{10}{\mu})} dy \, \| V(x_i + y) \|_{\ell_i^{2q}} (\mu h)^{\frac{d}{2q}} \\ &\leq h^{\frac{d}{2q}} \mu^{\frac{d}{q} - d} \| V \|_{2q} \, .\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{B_0(\frac{10}{\mu})} dy \int_{M_t \cap B_0(\frac{10}{R})} d\tau \int_{M_{t'} \cap B_0(\frac{10}{R})} d\tau' \, \mathbb{E} \max_{\mathcal{F}_k \times F_{k'}} |X_{\xi, \xi'}| \\ &\lesssim \sqrt{\ln(\mu R)} \cdot (2^k \vee 2^{k'})(2^{-k} \wedge 2^{-k'}) h^{\frac{d}{p'} + \frac{d}{2q}} \| V \|_{2q} \\ &= \sqrt{\ln(\mu R)} \cdot h^{\frac{d}{2}} \| V \|_{2q}\end{aligned}$$

Proof of Lemma 3 - Probabilistic estimate III

as desired.
(We used

$$\int_{M_t \cap B_0(\frac{10}{R})} d\tau \, \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{sc}}(\Lambda_R^*)}$$
$$\leq R^{\frac{d-1}{2}} \left(\int_{M_t \cap B_0(\frac{10}{R})} d\tau \, \|g(\eta_\nu + \tau)\|^2_{\ell^2_\nu(\Lambda_R^*)} \right)^{1/2} \left(\int_{M_t \cap B_0(\frac{10}{R})} d\tau \, 1 \right)^{1/2}$$
$$\lesssim \|g\|_{L^2(M_t)}$$

Proof of Lemma 3 - Deterministic estimate I

Recall $X_{\xi,\xi'} := \left|\sum_{j\in h\mathbb{Z}^d} \omega\,(j) \sum_i \mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)} V(x_i+y) \xi_i^{(k)} \xi_i^{(k')}\right|$. By Hölder

$$
\begin{aligned}
|X_{\xi,\xi'}| &\leq \sum_{j\in h\mathbb{Z}^d} \left| \sum_i V(x_i+y) \mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)} \xi_i^{(k)} \xi_i^{(k')} \right| \\
&\leq \sum_{j\in h\mathbb{Z}^d} \| V(x_i+y) \mathbb{1}_{x_i\in B_j(\frac{10}{\mu}+h)} \|_{\ell_i^1} \|\xi_i\|_{\ell_i^\infty} \|\xi_i'\|_{\ell_i^\infty} \,.
\end{aligned}
$$

Since

$$
\|\xi_i\|_{\ell_i^\infty} \|\xi_i'\|_{\ell_i^\infty} \leq 2^{-(k+k')} \|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{sc}}(\Lambda_R^*)} \|g'(\eta_{\nu'}+\tau')\|_{\ell^2_{\nu',\mathrm{sc}}(\Lambda_R^*)}
$$

Proof of Lemma 3 - Deterministic estimate II

and (using $|\{j \in h\mathbb{Z}^d : x_i + y \in Q_h + j\}| = 1$ for all i and Hölder),

$$\begin{aligned}\sum_{j \in h\mathbb{Z}^d} \|V(x_i+y)\mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)}\|_{\ell^1_i} &= \|V(x_i+y)\mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)}\|_{\ell^1_j \ell^1_i} \\ &= \|V(x_i+y)\mathbb{1}_{x_i \in B_j(\frac{10}{\mu}+h)}\|_{\ell^1_i \ell^1_j} = \|V(x_i+y)\|_{\ell^1_i} \\ &\le (R\mu)^{d-\frac{d}{2q}} \|V(x_i+y)\|_{\ell^{2q}_i} .\end{aligned}$$

Thus, we obtain (again by Hölder)

$$\begin{aligned}&\int_{B_0(\frac{10}{\mu})} dy \int_{M_t \cap B_0(\frac{10}{R})} d\tau \int_{M_{t'} \cap B_0(\frac{10}{R})} d\tau' \max_{\mathcal{F}_k \times F_{k'}} |X_{\xi,\xi'}| \\ &\lesssim (R\mu)^{d-\frac{d}{2q}} \mu^{\frac{d}{2q}-d} \|V\|_{2q} \cdot 2^{-(k+k')} .\end{aligned}$$

Local restriction theory

Local restriction theory I

$$\|(g\,d\sigma)^\vee\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}(M)} \Leftrightarrow \|(g\,d\sigma)^\vee\|_{L^{p'}(B_R(x_0))} \lesssim \|g\|_{L^{q'}(M)}$$

for all $x_0 \in \mathbb{R}^d$, $R > 0$, $g \in L^{q'}(M)$.
Localization in x-space induces a blurring in ξ-space. $\Rightarrow$ It suffices to consider the $1/R$-neighborhood $\mathcal{N}_{1/R}(M)$ of M. In fact,

$$\|(g\,d\sigma)^\vee\|_{L^{p'}(B_R(x_0))} \lesssim \|g\|_{L^{q'}(M)} \quad \text{for all } g \in L^{q'}(M)$$
$$\Leftrightarrow \|F^\vee\|_{L^{p'}(B_R(x_0))} \lesssim R^{-\frac{1}{q}} \|F\|_{L^{q'}(\mathcal{N}_{1/R}(M))} \quad \text{for all } F \in L^{q'}(\mathcal{N}_{1/R}(M))$$

Moreover, the uncertainty principle implies that F should be constant on $1/R$-balls.

Lemma 5 (Locally constant lemma)

For any $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$ *with* $\operatorname{supp}(f) \subseteq B_0(R)$ *we have* $\|\hat{f}\|_{L^\infty(B(\frac{1}{R}))} \lesssim \frac{\|\hat{f}\|_{L^1(w_{B(\frac{1}{R})})}}{|B(\frac{1}{R})|}$ *for any ball* $B(\frac{1}{R})$ *with radius* $\frac{1}{R}$*. Here* $w_{B(\frac{1}{R})}(\xi) = (1 + R \cdot \operatorname{dist}(\xi, B(\frac{1}{R})))^{-10d}$.

Local restriction theory II

Hence, it is natural to approximate F (the function on $\mathcal{N}_{1/R}(M)$ in Fourier space) by

$$F = \sum_{\eta \in \Lambda_R^*} F(\eta) \mathbb{1}_{B_\eta(R^{-1})}$$

where $\Lambda_R^* \subseteq M$ is a maximal $\frac{1}{R}$-separated subset.
Then, the continuous and discrete restriction estimates are equivalent to each other.

Definition 6

Let $\mathrm{Discres}'(M, p, q)$ denote the smallest number s.t. the following estimate holds for all $R \geq 2$, each collection $\Lambda_R^* \subseteq M$ of $\frac{1}{R}$-separated points, each sequence $a_\nu \in \mathbb{C}$, and each ball $B(R)$:

$$\| \sum_{\nu \in \Lambda_R^*} a_\nu \, e(\nu \cdot x) \|_{L^{p'}(B(R))} \leq \mathrm{Discres}'(M, p, q) R^{\frac{d-1}{q}} \| a_\nu \|_{\ell^{q'}(\Lambda_R^*)} .$$

Local restriction theory III

Definition 7

Similarly for $\mu \cdot R \geq 1$ let $\Lambda_\mu \subseteq B(R)$ denote a $\frac{1}{\mu}$-net and $\text{Discres}^{(\mu)}(M,p,q)$ denote the smallest number s.t.

$$\mu^{-\frac{d}{p'}} \| \sum_{\nu \in \Lambda_R^*} a_\nu \, e(\nu \cdot x) \|_{\ell^{p'}(\Lambda_\mu \cap B(R))} \leq \text{Discres}^{(\mu)}(M,p,q) R^{\frac{d-1}{q}} \|a_\nu\|_{\ell^{q'}(\Lambda_R^*)} \, .$$

Theorem 8

Let $1 \leq p, q \leq \infty$. Then
$\text{Discres}^{(\mu)}(M,p,q) \sim \text{Discres}'(M,p,q) \lesssim \|F_M^*\|_{L^{q'}(M) \to L^{p'}(\mathbb{R}^d)}$.

The (second) upper bound is standard, cf. Demeter's book.
To prove "$\sim$", we use among others

Lemma 9

Let $v \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} \hat{v} \subseteq B_0(1/h)$, $\Lambda_{h^{-1}}$ *be a set of h-separated points, and $p \geq 1$. Then*

$$h^{d/p} \|v\|_{\ell^p(\Lambda_{h^{-1}})} \lesssim \|v\|_{L^p(\mathbb{R}^d)} \, .$$

Covering numbers in Banach spaces

Covering numbers in Banach spaces

Recall $\Lambda_R^* \subseteq M_\lambda$ was a $\frac{1}{R}$-net with $\lambda R \geq 1$ and $\Lambda_\mu \subseteq B(R)$ was a $\frac{1}{\mu}$-net with $R\mu \geq 1$.
Recall the discrete extension operator

$$S : \ell^2_{\mathrm{sc}}(\Lambda_R^*) \to \ell^\infty(\Lambda_\mu \cap B(R)) ,$$
$$\{g(\eta_\nu + \tau)\}_\nu \mapsto \{ \sum_{\nu \in \Lambda_R^*} e((x_i + y)(\eta_\nu + \tau)) g(\eta_\nu + \tau)\}_i$$

where $x_i \in \Lambda_\mu$, $y \in B_0(\frac{10}{\mu})$, and $\tau \in B_0(\frac{10}{R})$.
Earlier we claimed the expansion

$$(Sg)_i = \sum_{k \geq 0} \xi_i^{(k)} , \quad \xi_i^{(k)} \in \mathcal{F}_k \subseteq \ell^\infty(\Lambda_\mu \cap B(R))$$

with

- $\ln |\mathcal{F}_k| < 4^k \lambda^{d-1} \ln(\mu R)$,
- $\|\xi_i^{(k)}\|_{\ell_i^\infty} \lesssim \lambda^{(-1+\frac{d}{1})/2} \cdot 2^{-k} \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{sc}}(\Lambda_R^*)}$, and
- $\|\xi_i^{(k)}\|_{\ell_i^{p'}} \lesssim \lambda^{(-1+\frac{d}{q})/2} \cdot \mu^{d/p'} \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{sc}}(\Lambda_R^*)}$.

Dual to Sudakov

Let $B_n^2 := \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ (euclidean unit ball) and let $\|\cdot\|_X$ be another (semi-)norm on $\mathbb{R}^n$ with corresponding unit ball $B_n^X = \{x \in \mathbb{R}^n : \|x\|_X \leq 1\}$. Define the **covering number**

$$N(B_n^2, B_n^X, t) = \min\{k \in \mathbb{N} | \exists (x_i)_{i=1}^k : B_n^2 \subseteq \bigcup_{i=1,\dots,k} (x_i + tB_n^X)\}.$$

Example: $N(B_n^2, B_n^2, t) \sim t^{-n}$ when $t < 1$.

Theorem 10 (Pajor, Tomczak–Jaegermann ('86) / Pajor, Talagrand (in Bourgain, Lindenstrauss, Milman) ('89))

Let

$$A_r := \int_{\mathbb{S}^{n-1}} \|x\|_X \, d\mu(x) \sim n^{-\frac{1}{2}} \mathbb{E} \| \sum_{j=1}^n g_j e_j \|_X ,$$

where g_j are gaussian rv's and $\{e_j\}_{j=1}^n$ denotes the standard basis in $\mathbb{R}^n$. Then

$$\ln N(B_n^2, B_n^X, t) \lesssim t^{-2} \cdot n \cdot A_r^2.$$

Application of dual to Sudakov I

Let $\|x\|_X := \|Sx\|_{\ell_m^\infty}$ for a linear map $S : \ell_n^2 \to \ell_m^\infty$. By Dudley's inequality,

$$
\begin{aligned}
A_r &= cn^{-\frac{1}{2}} \mathbb{E}_\omega \max_{k=1,\dots,m} \left| (S[\sum_j g_j e_j])(k) \right| \\
&\lesssim n^{-\frac{1}{2}} \sqrt{\ln m} \sup_{k=1,\dots,m} \left(\sum_j |(Se_j)(k)|^2 \|g_j\|_{\psi_2}^2 \right)^{\frac{1}{2}} \\
&\leq n^{-\frac{1}{2}} \sqrt{\ln m} \|S\|_{\ell_n^2 \to \ell_m^\infty} .
\end{aligned}
$$

Thus, B_n^2 can be covered by N many tB_n^X-balls with

$$
\ln N(B_n^2, B_n^X, t) \lesssim t^{-2} \cdot n \cdot n^{-\frac{2}{2}} (\ln m)^{\frac{2}{2}} \|S\|_{\ell^2 \to \ell^\infty}^2 = t^{-2} \ln(m) \|S\|_{\ell_n^2 \to \ell_m^\infty}^2
$$

and the right side is **independent of** n.
Equivalently, since $\|x\|_X = \|Sx\|_{\ell_m^\infty}$, we can cover $\{Sx : \|x\|_{\ell_n^2} \leq 1\}$ by N many tB_m^∞ balls.

Application of dual to Sudakov II

In the application $S : \ell^2_{sc}(\Lambda^*_R) \to \ell^\infty(\Lambda_\mu \cap B(R))$. Thus, $m \sim (R\mu)^d$ and $\|S\|_{\ell^2_{sc}(\Lambda^*_R)\to\ell^\infty(\Lambda_\mu\cap B(R))} \lesssim \lambda^{(-1+d)/2}\mu^{d/\infty}$.

Now let us cover $\{Sx : \|x\|_{\ell^2_n} \leq 1\}$ by $\bigcup_{j=1}^{N_j} B_{\phi_j}(t_j)$ where $t_j = 2^{-j}$ and $\ln(N_j) \lesssim t_j^{-2}\ln(\mu R)\lambda^{d-1}$. We collect the centers ϕ_j of these balls in the family/net $\mathcal{E}_j \subseteq \ell^\infty(\Lambda_\mu \cap B(R))$. Then

$$\max_{\|x\|\leq 1}\min_{\phi_j\in\mathcal{E}_j} \|Sx - \phi_j\|_{\ell^\infty} \leq t_j$$

or equivalently there is $\eta_j \in \ell^\infty(\Lambda_\mu \cap B(R))$ with $\|\eta_j\|_\infty < t_j$ such that

$$Sx = \phi_j + \eta_j\,.$$

Now telescope, i.e.,

$$Sx = \phi_k + \eta_k = \phi_0 - \phi_0 + \phi_1 - \phi_1 + \phi_2 - \phi_2 + ... - \phi_{k-1} + \phi_k + \eta_k\,, \quad \phi_j \in \mathcal{E}_j\,,$$

where ϕ_k depends on Sx. Thus,

$$\|Sx - (\phi_0 + \phi_1 - \phi_0 + \phi_2 - \phi_1 + ... + \phi_k - \phi_{k-1})\|_\infty = \|\eta_k\|_\infty < 2^{-k}$$

and the right side vanishes as $k \to \infty$.

Application of dual to Sudakov III

In particular, we pick the ϕ_j above such that the difference $\|\phi_j - \phi_{j-1}\|_\infty \leq 2^{-j} + 2^{1-j}$ (e.g., by taking ϕ_j such that $\|Sx - \phi_j\| \leq 2^{-j}$). (Thus, ϕ_k depends on Sx; ϕ_{k-1} depends on Sx and ϕ_k; and so on.)

Collecting *all vectors* $\xi^{(k)} = \phi_k - \phi_{k-1}$ for which $\|\xi^{(k)}\| \leq 2^{-k} + 2^{1-k}$, we may thus obtain sets $\mathcal{F}_k \subseteq \mathcal{E}_k - \mathcal{E}_{k-1} = \{\phi_k - \phi_{k-1} : \phi_k \in \mathcal{E}_k, \phi_{k-1} \in \mathcal{E}_{k-1}\}$ for $k \geq 1$ and $\mathcal{F}_0 = \mathcal{E}_0$, and obtain, for any $x \in \ell_n^2$, the expansion

$$Sx = \sum_{k \geq 0} \xi^{(k)} \quad \text{for } \xi^{(k)} \in \mathcal{F}_k \subseteq \ell^\infty(\Lambda_\mu \cap B(R))$$

with $\|\xi^{(k)}\|_\infty \lesssim 2^{-k} \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{sc}}(\Lambda_R^*)}$. By $|\mathcal{F}_k| \leq |\mathcal{E}_k| \cdot |\mathcal{E}_{k-1}|$ and dual-to-Sudakov, we have

$$\ln|\mathcal{F}_k| \leq \ln|\mathcal{E}_k| + \ln|\mathcal{E}_{k-1}| \lesssim 4^k \ln(R\mu) \cdot \lambda^{d-1}\,.$$

Finally,

$$\|\xi^{(k)}\|_{\ell^{p'}(\Lambda_\mu \cap B(R))} \lesssim \lambda^{(-1+\frac{d}{q})/2} \mu^{d/p'} \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{sc}}(\Lambda_R^*)}$$

is a consequence of

$$\|S\|_{\ell^2_{\mathrm{sc}}(\Lambda_R^*) \to \ell^{p'}(\Lambda_\mu \cap B(R))} \lesssim \lambda^{(-1+\frac{d}{q})/2} \mu^{d/p'}$$