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A topological index for one-dimensional quantum walks with asymptotically periodic parameters

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Motivation

⚠ What exactly is a **(discrete-time) quantum walk**??

- Let $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ be the Hilbert space of square-summable $\mathbb{C}^2$-valued sequences.
- A (discrete-time) ***2-state quantum walk on the integer lattice*** $\mathbb{Z}$ is characterised by a unitary ***time-evolution operator*** U on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$.
  - Suppose that a ***(quantum) walker*** has an initial state $\Psi_0 = (\Psi_0(x))_{x \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, where we assume $\|\Psi_0\|_{\ell^2(\mathbb{Z},\mathbb{C}^2)} = 1$.
  - Then the ***state*** of the walker at time $t \in \{1, 2, \dots\}$ is given by $U^t \Psi_0$.
  - The probability of finding the walker at position $x \in \mathbb{Z}$ and at time $t \in \{1, 2, \dots\}$ is given by $P_t(x) := \|(U^t \Psi_0)(x)\|^2_{\mathbb{C}^2}$.
  - Conservation of total probability:

$$\sum_{x \in \mathbb{Z}} P_t(x) = \sum_{x \in \mathbb{Z}} \|(U^t \Psi_0)(x)\|^2_{\mathbb{C}^2} = \|U^t \Psi_0\|^2_{\ell^2(\mathbb{Z},\mathbb{C}^2)} = \|\Psi_0\|^2_{\ell^2(\mathbb{Z},\mathbb{C}^2)} = 1, \quad t \in \{1, 2, \dots\}.$$

- We consider **index theory** for unitary time-evolution operators of the form;

$$\boxed{U = \varGamma \times \varGamma',}$$

where $\varGamma, \varGamma'$ are **unitary self-adjoint** operators on a state Hilbert space $\mathcal{H}$.

- Given $U = \varGamma \times \varGamma'$, we get the following **chiral symmetry condition**:

$$U^* =$$

Aim of the talk

Let λ be a fixed number in the unit-circle $\mathbb{T}$. We wish to introduce a certain well-defined **index**, say $\mathrm{ind}_\lambda(\varGamma, U)$, satisfying:

(i) $\mathrm{ind}_\lambda(\varGamma, U)$ is **stable** against a wide range of perturbations.

(ii) $|\mathrm{ind}_\lambda(\varGamma, U)| \leq \dim\ker(U - \lambda)$ (known as **symmetry protection of eigenstates**).

⚠ We are interested in the case $\lambda = \pm 1$.

Preliminaries

- Let Γ be a (bounded) operator on an abstract Hilbert space $\mathcal{H}$:
  - Γ is **self-adjoint**, if $\Gamma^* = \Gamma$.
  - Γ is **unitary**, if $\Gamma^* = \Gamma^{-1}$.
  - Γ is **involutory**, if $\Gamma^2 = 1$.
- If Γ has any two of the above properties, then it automatically has the third property.
- For example, the following 2×2 matrices are unitary self-adjoint;

$$\Gamma_a := \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}, \qquad a \in [-1, 1].$$

- If $\Gamma : \mathcal{H} \to \mathcal{H}$ is unitary self-adjoint, then:
  - We have $\mathcal{H} = \ker(\Gamma - 1) \oplus \ker(\Gamma + 1)$. Indeed, for any $\Psi \in \mathcal{H}$

$$\Psi = \left(\frac{1+\Gamma}{2}\right)\Psi + \left(\frac{1-\Gamma}{2}\right)\Psi \in \ker(\Gamma - 1) \oplus \ker(\Gamma + 1).$$

  - The operator Γ has the following block-operator matrix representation;

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\ker(\Gamma-1)\oplus\ker(\Gamma+1)} =$$

⚠ Indeed, if $\Psi_\pm \in \ker(\Gamma \mp 1)$, then $\Gamma\Psi_\pm = \pm\Psi_\pm$.

- A decomposition of this kind is often used, for example, in **supersymmetric quantum mechanics (SUSYQM)**.

Index Theory for Chiral Unitaries

- Let $U = \varGamma \times \varGamma'$, where $\varGamma, \varGamma'$ are unitary self-adjoint operators on $\mathcal{H}$.
  - The operator $R := (U + U^*)/2$, which is the **real part** of U, satisfies

$$\ker(U \mp 1) = \ker(R \mp 1). \tag{1}$$

  - We can also easily prove that R can be written as

$$R = R_1 \oplus R_2 \quad \text{w.r.t.} \ \ker(\varGamma - 1) \oplus \ker(\varGamma + 1). \tag{2}$$

- It immediately follows from (1) and (2) that $\ker(U \mp 1) = \ker(R_1 \mp 1) \oplus \ker(R_2 \mp 1)$.
- This motivates us to introduce the following two formal indices:

$$\boxed{\operatorname{ind}_{\pm}(\varGamma, U) := \dim\ker(R_1 \mp 1) - \dim\ker(R_2 \mp 1),}$$

  where $|\operatorname{ind}_{\pm}(\varGamma, U)| \leq \dim\ker(U \mp 1)$.
- The formal index $\operatorname{ind}_{\pm}(\varGamma, U)$ is well-defined, if $\dim\ker(U \mp 1) < \infty$.

- Proof of $\ker(U \mp 1) = \ker(R \mp 1)$ (equality (1) on the previous slide):

- Proof of $R = R_1 \oplus R_2$ (equality (2) on the previous slide):

  ⚠ Let us first show that $U^* = \Gamma U \Gamma$ implies $R = \Gamma R \Gamma$ (that is, $[R, \Gamma] = 0$).

- The operator $Q := (U - U^*)/2i$, which is the **imaginary part** of U, satisfies

$$Q = \begin{pmatrix} 0 & Q_0^* \\ Q_0 & 0 \end{pmatrix}_{\ker(\Gamma-1)\oplus\ker(\Gamma+1)},$$

because of the anti-commutation relation $\Gamma Q + Q\Gamma = 0$.

- We define the following formal index

$$\boxed{\operatorname{ind}(\Gamma, U) := \dim\ker Q_0 - \dim\ker Q_0^*,}$$

which is the **Fredholm index** of Q_0, provided that Q_0 is Fredholm. Moreover,

$$\operatorname{ind}(\Gamma, U) = \operatorname{ind}_+(\Gamma, U) + \operatorname{ind}_-(\Gamma, U), \tag{3}$$

$$|\operatorname{ind}(\Gamma, U)| \leq \dim\ker(U - 1) + \dim\ker(U + 1), \tag{4}$$

where (4) is a weaker version of $|\operatorname{ind}_\pm(\Gamma, U)| \leq \dim\ker(U \mp 1)$ previously mentioned.

A Concrete Model

- We consider the following block-operator matrices on $\ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}, \mathbb{C}) \oplus \ell^2(\mathbb{Z}, \mathbb{C})$:

$$\varGamma := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \times \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}, \tag{5}$$

$$\varGamma' := \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}. \tag{6}$$

  where:

  - L is the **left-shift operator** on $\ell^2(\mathbb{Z}, \mathbb{C})$.
  - $p = (p(x))_{x\in\mathbb{Z}}$ and $a = (a(x))_{x\in\mathbb{Z}}$ are two arbitrary sequences taking values in $[-1, 1]$.
- Note that $\varGamma, \varGamma'$ are unitary self-adjoint.
- Let $U := \varGamma \times \varGamma'$ (time-evolution of the so-called **split-step quantum walk**).
- Let $\sigma_{\mathrm{ess}}(U)$ be the **essential spectrum** of U. If $\pm 1 \notin \sigma_{\mathrm{ess}}(U)$, then $\ker(U \mp 1)$ is finite-dimensional, and so $\mathrm{ind}_{\pm}(\varGamma, U)$ is well-defined.

Theorem 1. Symmetry Protection of Eigenstates for the SSQW

Let us assume the existence of the following limits for each $\star = -\infty, +\infty$:

$$p(\star) := \lim_{x \to \star} p(x), \qquad a(\star) := \lim_{x \to \star} a(x). \tag{7}$$

Then $\pm 1 \notin \sigma_{\mathrm{ess}}(U)$ *if and only if* $p(\star) \mp a(\star) \neq 0$ *for each* $\star = -\infty, +\infty$. *In this case,*

(i) **Index Formula.** *We have* $\mathrm{ind}_{\pm}(\Gamma, U) \in \{-1, 0, 1\}$. *More precisely,*

$$\mathrm{ind}_{\pm}(\Gamma, U) = \frac{\mathrm{sign}\,(p(+\infty) \mp a(+\infty)) - \mathrm{sign}\,(p(-\infty) \mp a(-\infty))}{2}, \tag{8}$$

where sign *denotes the sign function.*

(ii) **Symmetry Protection.** *If* $\sup_{x \in \mathbb{Z}} |\zeta(x)| < 1$ *for each* $\zeta = p, a$, *then*

$$|\mathrm{ind}_{\pm}(\Gamma, U)| = \dim \ker(U \mp 1). \tag{9}$$

⚠ The equality (9) can be viewed as an analogue of the **bulk-edge correspondence**.

⚠ Let $\mathbb{T}$ be the unit-circle.

- It can be shown that for each $\star = -\infty, +\infty$ there exist **continuous** $\mathbb{T} \ni z \longmapsto \gamma_\pm(z, \star) \in \mathbb{C}$ with the following properties:
  - The continuous curve $\gamma_\pm(\cdot, \star)$ depends only on $p(\star), a(\star)$.
  - The integer $\mathrm{ind}_\pm(\Gamma, U)$ is a **topological index** in the sense that

$$\mathrm{ind}_\pm(\Gamma, U) = \mathrm{wn}(\gamma_\pm(\cdot, +\infty)) - \mathrm{wn}(\gamma_\pm(\cdot, -\infty)), \tag{10}$$

    where wn denotes the **winding number**.
- Therefore, the equality $|\mathrm{ind}_\pm(\Gamma, U)| = \dim\ker(U \mp 1)$ previously mentioned can be understood as the **bulk-edge correspondence** for the one-dimensional SSQW.
- The non-trivial equality (10) can be proved by the well-known index theorem for **Toeplitz operators**.

- The Hilbert space $L^2(\mathbb{T})$ admits the standard complete orthonormal basis $(e_x)_{x \in \mathbb{Z}}$ defined by $\mathbb{T} \ni z \longmapsto z^x \in \mathbb{C}$.
- The **Hardy-Hilbert space** H^2 is the closure of the linear span of $\{e_x \mid x \geq 0\}$.
- Let $\iota : \mathrm{H}^2 \hookrightarrow L^2(\mathbb{T})$ be the inclusion mapping, and let $f \in C(\mathbb{T})$. Then the **Toeplitz operator** T_f with symbol f is defined by

$$\boxed{T_f := \iota^* \times M_f \times \iota,} \tag{11}$$

where $M_f : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the bounded **multiplication operator** by f.

  - The Toeplitz operator T_f is Fredholm if and only if the curve $\mathbb{T} \ni z \longmapsto f(z) \in \mathbb{C}$ does NOT pass through the origin. In this case, $\mathrm{ind}\, T_f = -\mathrm{wn}(f)$.
  - The essential spectrum of T_f is given by $\sigma_{\mathrm{ess}}(T_f) = \{f(z) \mid z \in \mathbb{T}\} = \mathrm{ran}\, F$.

Two Generalisations of Theorem 1

- What follows is the main result of **arXiv:2111.04108**. This is a joint work with
  - Y. Matsuzawa (Shinshu University), A. Suzuki (Shinshu University), N. Teranishi (Hokkaido University), K. Wada (Hachinohe Kosen).
- In Theorem 1, we have

$$\boxed{\pm 1 \notin \sigma_{\mathrm{ess}}(U) \iff p(\star) \mp a(\star) \neq 0 \text{ for each } \star = -\infty, +\infty.}$$

- We say that U is **gap-less**, if the above condition **fails to hold.** In this case, scattering theory for supersymmetric quantum mechanics allows us to show

$$\mathrm{ind}_{\pm}(\Gamma, U) = \frac{\mathrm{sign}\,(p(+\infty) \mp a(+\infty)) - \mathrm{sign}\,(p(-\infty) \mp a(-\infty))}{2},$$

where we now agree to set $\mathrm{sign}\,(0) := 0$. Therefore, $\mathrm{ind}_{\pm}$ can take **half-integer values:**

$$\mathrm{ind}_{\pm}(\Gamma, U) \in \left\{-1, \frac{-1}{2}, 0, \frac{1}{2}, 1\right\}.$$

- What follows is the main result of **arXiv:2111.12652**. This is a joint work with
  - Y. Matsuzawa (Shinshu University) and K. Wada (Hachinohe Kosen).
- In Theorem 1, we assume the existence of the following two-sided limits:

$$\zeta(\pm\infty) := \lim_{x\to\pm\infty} \zeta(x), \qquad \zeta = p, a. \tag{12}$$

  We can obtain a generalisation of Theorem 1, if we replace (12) by the so-called **"asymptotically periodic assumption"**.
- For example, in the "asymptotically 2-periodic case", we assume the existence of

$$\zeta(\star, 0) := \lim_{x\to\star} \zeta(2x+0),$$
$$\zeta(\star, 1) := \lim_{x\to\star} \zeta(2x+1).$$

- If you are interested, I am happy to give you more details about this.

⚠ This talk is based on the following existing literature:

- J. K. Asbóth, H. Obuse, Bulk-boundary correspondence for chiral symmetric quantum walks, Phys. Rev. B **88**(12), 121406 (2013).
- C. Cedzich et al., Bulk-edge correspondence of one-dimensional quantum walks, J. Phys. A **49**(21), 21LT01 (2016).
- C. Cedzich et al., The Topological Classification of One-Dimensional Symmetric Quantum Walks, Ann. Henri Poincaré **19**(2), 325–383 (2018).
- C. Cedzich et al., Complete homotopy invariants for translation invariant symmetric quantum walks on a chain, Quantum **2**, 95 (2018).
- T. Fuda, D. Funakawa, A. Suzuki, Localization for a one-dimensional split-step quantum walk with bound states robust against perturbations, J. Math. Phys. **59**(8), 082201 (2018).
- A. Suzuki, Supersymmetry for chiral symmetric quantum walks, Quantum Inf. Process. **18**(12), 363 (2019).
- Y. Matsuzawa, An index theorem for split-step quantum walks, Quantum Inf. Process. **19**(8), 227 (2020).

My slides can be found in