The asymptotic expansions of the hypergeometric function with respect to a parameter

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Gauss HGDE:

$$x(1-x)\frac{d^2w}{dx^2} + (c-(a+b+1)x)\frac{dw}{dx} - abw = 0 \ (a,\ b,\ c \in \mathbb{C}). \quad (1)$$

- **regular singular points $0, 1, \infty$.**
- **The hypergeometric series (or function): $(c \neq 0, -1, -2, \dots)$**

$$_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \quad (2)$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$, etc.

- **$_2F_1(a,b,c;x)$ is convergent in the open unit disk with the center at the origin in the complex plane.**

- **Kummer's solutions of** (1)

$$\begin{cases} u_1 &= {}_2F_1(a,b,c;x), \\ u_2 &= {}_2F_1(a,b,a+b+1-c;1-x), \\ u_3 &= (-x)^{-a}\,{}_2F_1(a,a+1-c,a+1-b;\frac{1}{x}), \\ u_4 &= (-x)^{-b}\,{}_2F_1(b,b+1-c,b+1-a;\frac{1}{x}), \\ u_5 &= x^{1-c}\,{}_2F_1(a-c+1,b-c+1,1-c;x), \\ u_6 &= (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b,c-a-b+1;1-x). \end{cases}$$

Kummer's solutions are expressed in the hypergeometric function.
(u_1, u_5) **: fundamental solutions in the neighborhood of** $x = 0$.
(u_2, u_6) **: fundamental solutions in the neighborhood of** $x = 1$.
(u_3, u_4) **: fundamental solutions in the neighborhood of** $x = \infty$.
In this talk, we consider asymptotic expansions of u_1 **and** u_5 **with respect to a parameter, respectively.**

The asymptotic expansion of ${}_2F_1$ with respect to the parameter

There are more previous studies.

[I] K., Iwasaki, On Some Hypergeometric Summations II. Duality and Reciprocity, e-Print arXiv: 1504.0314v2.

[O] A.B, Olde Daalhuis, Uniform asymptotic expansions for hypergeometric functions with large parameters. II. *Analysis and Applications*, (*S*ingapore) 1(1) (2003), 121– 128.

[OLBC] F.W.J., Olver, D.W., Lozier, R.F. Boisvert, and C.W., Clark, (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.

[P] R,B, Paris, Asymptotics of the Gauss hypergeometric function with large parameters, I, *Journal of Classical Analysis*, 2 (2013), 183–203.

[W] G. N. Watson, Asymptotic expansions of hypergeometric functions,Trans. Cambridge Philos. Soc. 22 (1918), 277-308.

etc.

Example

In this book, Professor Daalhuis introduces a large parameter η by setting: $a = \alpha_0, b = \beta_0, c = \gamma_0 + \eta$ and $\eta \to +\infty$, u_1 has the following form:

$$ {}_2F_1(\alpha_0, \beta_0, \gamma_0 + \eta; x) \sim \frac{\Gamma(\eta + \gamma_0)}{\Gamma(\gamma_0 - \beta_0 + \eta)} \sum_{k=0}^{\infty} q_s(x)(\beta_0)_s \eta^{-s-\beta_0}. $$

Here $q_0(x) = 1$ and $q_s(x)$ ($s = 1, 2, \cdots$) are defined by the generating function:

$$ (\frac{e^t - 1}{t})^{\beta_0 - 1} e^{t(1-\gamma_0)} (1 - x + xe^{-t})^{-a} = \sum_{s=0}^{\infty} q_s(x) t^s. $$

In this talk, we present the asymptotic expansions for the parameters of u_1 and u_5 from the viewpoint of the exact WKB analysis for case that has not been proved in previous results, namely, the case where $a = \alpha_0 + \eta\alpha, b = \beta_0 + \eta\beta, c = \gamma_0 + \eta\gamma$ ($\alpha_0, \beta_0, \gamma_0, \alpha, \beta, \gamma \in \mathbb{C}$), albeit conditionally.

•The exact WKB analysis $\cdots$ Global analysis in the complex domain using Borel sums of formal solutions.

⋄ What is the Borel sum (Borel summable) ?

[KT] Kawai-Takei, Algebraic analysis of singular perturbation theory, AMS, 2005.

For a formal series with an exponential term ($\alpha \in \mathbb{R} - \{0, -1, -2, \dots\}$):

$$\varphi(x, \eta) = \exp(\eta a(x)) \sum_{n=0}^{\infty} \varphi_j(x) \eta^{-\alpha-n},$$

($a(x)$, $\varphi_j(x)$: holomorphic function in an open set $U \subset \mathbb{C}$).

a Borel transform of $\varphi(x, \eta)$ at η is defined by

$$\varphi_B(x, y) := \sum_{n=0}^{\infty} \frac{\varphi_j(x)}{\Gamma(\alpha + n)} (y + a(x))^{\alpha+n-1}.$$

For φ to be Borel summable in U, the following holds.

⋄ **For any compact set $U' \subset U$, $R > 0$ exists, φ_B converges in $\{(x, y) \,|\, x \in U, \ |y + a(x)| < R\}$.**

⋄ **For any compact set $U' \subset U$, $R > 0$ exists, φ_B can be analytically continued to the following set**

$$\{(x, y) \,|\, x \in U, \ |\mathrm{Im}\,(y + a(x))| < R, \ \mathrm{Re}\,(y + a(x)) > 0\}$$

and constant $\exists M > 0$, a integral

$$\Phi(x, \eta) := \int_{-a(x)}^{\infty} \varphi_B(x, y) \exp(-\eta y) dy$$

converge for $\forall x \in U$ and $\forall \eta \geq M$ and put a analytic function.

$\Phi(x, \eta)$ is called a Borel sum of φ.

Watson's Lemma ([W2]):

The Borel sum $\Phi(x, \eta)$ has an asymptotic expansion:

$$\Phi(x, \eta) \sim \varphi(x, \eta) \quad \text{as } \eta \to +\infty.$$

Outline

- **If we introduce a large parameter η in the hypergeometric differential equation suitably, we can construct the WKB solutions of the equation.**
- **The WKB solutions normalized suitably are Borel summable and the Borel sums are analytic solutions to the equation.**
- **Using Watson's Lemma, the Borel sums of the WKB solutions have asymptotic expansions of the WKB solutions with respect to the parameter.**
- **The hypergeometric function can be expressed explicitly as a linear combination of the Borel resummed the WKB solutions.**

$$(u_1, u_5) = (p(x)\Psi_+, p(x)\Psi_-)A$$

A: A regular matrix,
$p(x)$: a power of x function,
$\Psi_\pm$: Borel sums of the WKB solutions,
$u_1 = {}_2F_1(a, b, c; x)$, $u_5 = x^{1-c}\,{}_2F_1(a - c + 1, b - c + 1, 1 - c; x)$.

Using the above relation, we obtain the asymptotic expansions of u_1, u_5 with respect to the parameter from viewpoint of the exact WKB analysis.

We recall the Gauss hypergeometric differential equation:

$$x(1-x)\frac{d^2w}{dx^2}+(c-(a+b+1)x)\frac{dw}{dx}-abw=0. \tag{1}$$

- **Introduce a large parameter η by setting:**

$$a=\alpha_0+\eta\alpha,\ b=\beta_0+\eta\beta,\ c=\gamma_0+\eta\gamma.\ (\alpha_0,\beta_0,\gamma_0,\alpha,\beta,\gamma\in\mathbb{C})$$

- **Eliminate the first-order term by setting:**

$$\psi=x^{(\gamma_0+\eta\gamma)/2}(1-x)^{(\alpha_0+\beta_0-\gamma_0+1+\eta(\alpha+\beta-\gamma))/2}w.$$

- **The Schrödinge equation with the large parameter η:**

$$\Big(-\frac{d^2}{dx^2}+\eta^2 Q\Big)\psi=0. \tag{3}$$

Here $Q=Q_0+\eta^{-1}Q_1+\eta^{-2}Q_2$,

$$Q_0=\frac{(\alpha-\beta)^2x^2+2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2},$$

$$Q_1=\frac{(\alpha-\beta)(\alpha_0-\beta_0)x^2+(2(\alpha_0\beta+\alpha\beta_0)-\beta\gamma_0-\beta_0\gamma-\alpha\gamma_0-\alpha_0\gamma+\gamma)x+\gamma(\gamma_0-1)}{2x^2(x-1)^2},$$

$$Q_2=\frac{(\alpha_0-\beta_0+1)(\alpha_0-\beta_0-1)x^2+2(2\alpha_0\beta_0-\beta_0\gamma_0-\alpha_0\gamma_0+\gamma_0)x+\gamma_0(\gamma_0-2)}{4x^2(x-1)^2}.$$

Some base notation of the Exact WKB analysis

• **WKB solutions normalized at a:**

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\pm \int_a^x S_{\mathrm{odd}}\, dx \Big).$$

• $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j = S_{\mathrm{odd}} + S_{\mathrm{even}}$ **is a formal solution to Riccati equation**

$$\frac{dS}{dx} + S^2 = \eta^2 Q$$

associated with (3)**.**

• **A turning point a:**

A zero of $Q_0 dx^2$ is called a turning point of (3)**.**
(If a simple zero of $Q_0 dx^2$, we call a simple turning point of (3)**.)**
Our equation (3) **has two turning points $a = a_0, a_1$.**

• **We choice the leading term $S_{-1} = \sqrt{Q_0}$. ($S_{-1}^2 = Q_0$)**

•**A Stokes curve of emanating from the turning point:**

An integral curve of

$$\mathrm{Im} \sqrt{Q_0} dx = 0$$

emanating from the turning point.

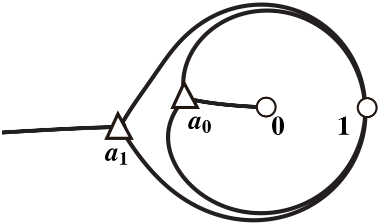
- **A Stokes region:**

A Region surrounded by Stokes curves.

- **A Stokes graph of our equation:**

A collection of all the Stokes curves, the turning points and the singular points.
In our equation, there are four types of the Stokes graphs.

Example of the Stokes graph



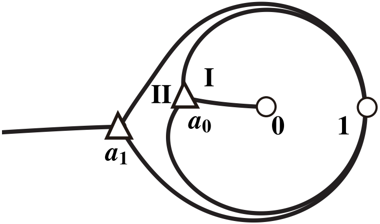
$0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta$

In this case, the Stokes geometry is non-degenerate, that is, there are not Stokes curves which connect turning point(s).

If the Stokes graph is not degenerate, $\psi_\pm$ are Borel summable on each Stokes region (Koike-Schäfke [KS]). The Borel sums of the WKB solutions $\psi_\pm$ are analytic solutions to (3).

The connection formula (Voros [V])

The WKB solutions $\psi_\pm$ have Stokes phenomena on these Stokes curves.



$0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta$

I and II: Stokes regions.

$\psi_\pm$: WKB solutions normalized at a_0.

$\Psi_\pm^{\mathrm{I}}$, $\Psi_\pm^{\mathrm{II}}$: Borel sums of $\psi_\pm$ in I and II, respectively.

If we cross the Stokes curve in a counterclockwise manner,i.e., from I to II, and $\mathrm{Re}\int_{a_0}^{x} \sqrt{Q_0}\,dx < 0$ on the Stokes curve, we have the following forms:

$$\begin{cases} \Psi_+^{\mathrm{I}} = \Psi_+^{\mathrm{II}}, \\ \Psi_-^{\mathrm{I}} = i\Psi_+^{\mathrm{II}} + \Psi_-^{\mathrm{II}}. \end{cases}$$

In this case, ψ_- is a dominant solution and ψ_+ is recessive solution. If the WKB solution is the recessive solution, it doesn't occur the Stokes phenomena.

Assumption

- $(\alpha, \beta, \gamma) \notin E_0 \cup E_1 \cup E_2$

$$E_0 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha \cdot \beta \cdot \gamma \cdot (\alpha - \beta) \cdot (\alpha - \gamma) \cdot (\beta - \gamma) \cdot (\alpha + \beta - \gamma) = 0\},$$

$$E_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\,\alpha\, \mathrm{Re}\,\beta\, \mathrm{Re}\,(\gamma - \alpha)\, \mathrm{Re}\,(\gamma - \beta) = 0\},$$

$$E_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\,(\alpha - \beta)\, \mathrm{Re}\,(\alpha + \beta - \gamma)\, \mathrm{Re}\,\gamma = 0\}.$$

$(\alpha, \beta, \gamma) \notin E_0$: There are two simple turning points a_0 and a_1 of (3).
$(\alpha, \beta, \gamma) \notin E_1 \cup E_2$: There are not Stokes curves which connect turning point(s).

- Take the branch of $\sqrt{Q_0}$ as $\sqrt{Q_0} \sim \dfrac{\gamma}{2x}$ near the origin.

We define

$$\omega_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\gamma < \mathrm{Re}\,\beta\},$$

$$\omega_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha + \mathrm{Re}\,\beta\},$$

$$\omega_3 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta\},$$

$$\omega_4 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha + \mathrm{Re}\,\beta < \mathrm{Re}\,\beta\}.$$

Characterization of the Stokes graphs with respect to parameters

Let ι_m $(m = 0, 1, 2)$ denote respectively the following involutions:

$$\iota_0 : (\alpha, \beta, \gamma) \mapsto (\beta, \alpha, \gamma),$$
$$\iota_1 : (\alpha, \beta, \gamma) \mapsto (\gamma - \alpha, \gamma - \beta, \gamma),$$
$$\iota_2 : (\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma).$$

Set

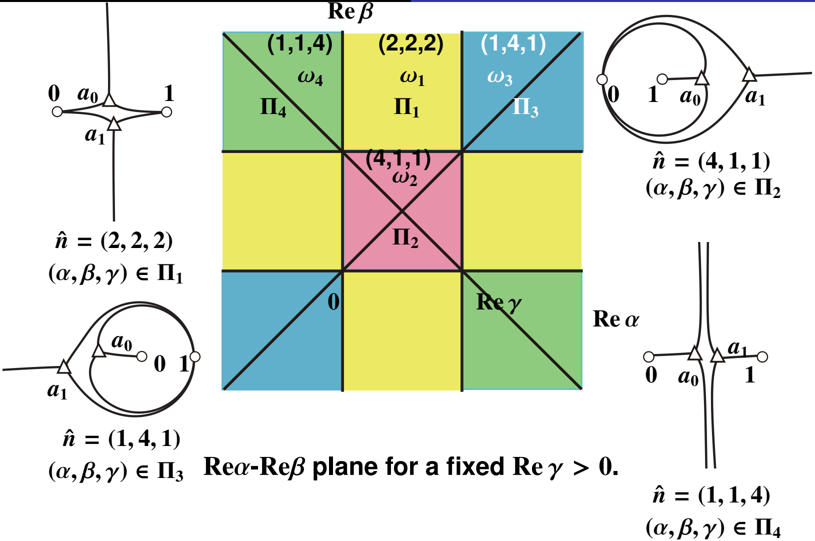
$$\Pi_k = \bigcup_{r \in G} r(\omega_k) \quad (k = 1, \ldots, 4).$$

- **Characterization of Stokes geometry in terms of (α, β, γ):**

Let n_0, n_1 and n_2 be the numbers of Stokes curves that flow into 0, 1 and ∞, respectively and let $\hat{n}$ denote (n_0, n_1, n_2).

Theorem 1(Aoki-T [AT]).

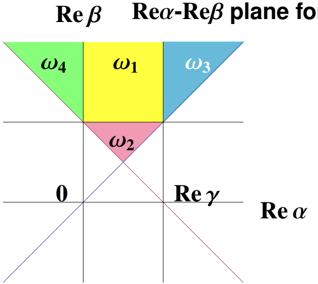
(1) $(\alpha, \beta, \gamma) \in \Pi_1 \to \hat{n} = (2, 2, 2)$. **(2)** $(\alpha, \beta, \gamma) \in \Pi_2 \to \hat{n} = (4, 1, 1)$.
(3) $(\alpha, \beta, \gamma) \in \Pi_3 \to \hat{n} = (1, 4, 1)$. **(4)** $(\alpha, \beta, \gamma) \in \Pi_4 \to \hat{n} = (1, 1, 4)$.



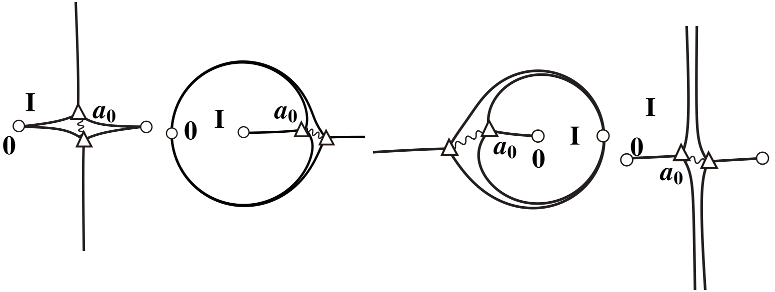
Reα-Reβ plane for a fixed Re $\gamma > 0$.

We consider the asymptotic expansions of u_1 and u_5 with respect to the parameter in the case where $(\alpha,\beta,\gamma) \in \omega_k$ ($k = 1,2,3,4.$).
There are the different results in each domain ω_k.

$\mathrm{Re}\alpha$-$\mathrm{Re}\beta$ plane for a fixed $\mathrm{Re}\,\gamma > 0$.



We investigate the asymptotic expansions of u_1 and u_5 with respect to the parameter in the following Stokes regions I, respectively.



$(\alpha, \beta, \gamma) \in \omega_1$ $(\alpha, \beta, \gamma) \in \omega_2$ $(\alpha, \beta, \gamma) \in \omega_3$ $(\alpha, \beta, \gamma) \in \omega_4$

$\psi_\pm$ denote the WKB solutions normalized at a_0. Let us recall that we take the branch of $\sqrt{Q_0}$ as $\sqrt{Q_0} \sim \gamma/2x$ near the origin. In this case, ψ_+ is the recessive solution and ψ_- is the dominant solution.

- **$\Psi^{\mathrm{I}}_\pm$ denote the Borel sums of $\psi_\pm$ in I.**

There is a regular matrix A_j $(j = 1, 2, 3, 4)$ such that

$$(u_1, u_5) = (p(x)\Psi^{\mathrm{I}}_+, p(x)\Psi^{\mathrm{I}}_-)A_j$$

and it is possible to obtain the explicit form of A_j. Here, $p(x) = x^{-c/2}(1-x)^{-(a+b-c+1)/2}$, $u_1 = {}_2F_1(a, b, c; x)$ and $u_5 = x^{1-c}\, {}_2F_1(a-c+1, b-c+1, 1-c; x)$.
Using the above relation and Watson's Lemma [W2], we get the following theorem:

Theorem 2 (Aoki-Takahashi-T[ATT1])

If $(\alpha, \beta, \gamma) \in \omega_j$ and $\eta \to \infty$, (u_1, u_5) have the following asymptotic expansions near the origin with respect to the parameter:

$$(u_1, u_5) \sim (p(x)\psi_+, p(x)\psi_-)A_j \quad (j = 1, 2, 3, 4),$$

where

$$p(x) = x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}(a = \alpha_0 + \eta\alpha, b = \beta_0 + \eta\beta, c = \gamma_0 + \eta\gamma).$$

(i) The regular matrix A_1 in ω_1 has the following form:

$$A_1 = \begin{pmatrix} \sqrt{\frac{c-1}{2}} \exp \hat{V}_0^1 & -i\frac{\exp(2\pi i a)-\exp(2\pi i c)}{\exp(2\pi i c)-1} \sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^1) \\ 0 & \sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^1) \end{pmatrix}.$$

Here

$$\hat{V}_0^1 = \frac{1}{2} \log \frac{\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)} \exp\left(\pi i \left(a - \frac{1}{2}\right)\right).$$

(ii) The regular matrix A_2 in ω_2 has the following form:

$$A_2 = \begin{pmatrix} \sqrt{\frac{c-1}{2}} \exp \hat{V}_0^2 & i\frac{(\exp(2\pi i(a-c))-1)(\exp(2\pi i(b-c))-1)\exp(2\pi i c)}{\exp(2\pi i c)-1} \sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^2) \\ 0 & \sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^2) \end{pmatrix}.$$

Here

$$\hat{V}_0^2 = \frac{1}{2} \log \frac{2\pi\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \exp(\pi i(a+b-c)).$$

(iii) The regular matrix A_3 in ω_3 has the following form:

$$A_3 = \begin{pmatrix} \sqrt{\frac{c-1}{2}} \exp \hat{V}_0^3 & i\frac{\exp(2\pi i c)}{\exp(2\pi i c)-1}\sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^3) \\ 0 & \sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^3) \end{pmatrix}.$$

Here

$$\hat{V}_0^3 = \frac{1}{2} \log \frac{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)}{2\pi\Gamma(a)\Gamma(b)} \exp(\pi i(1-c)).$$

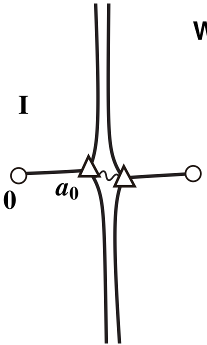
(iv) The regular matrix A_4 in ω_4 has the following form:

$$A_4 = \begin{pmatrix} \sqrt{\frac{c-1}{2}} \exp \hat{V}_0^4 & \frac{i}{\exp(2\pi i c)-1}\sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^4) \\ 0 & \sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^4) \end{pmatrix}.$$

Here

$$\hat{V}_0^4 = \frac{1}{2} \log \frac{\Gamma(1-a)\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)}{2\pi\Gamma(b)\Gamma(c-a)}.$$

Outline of proof (We only show the proof in the case where ω_4.)



We recall that $\Psi^{\mathrm{I}}_{\pm}$ are the Borel sums of $\psi_{\pm}$ in I.
Firstly, we consider the asymptotic expansion of u_1 with respect to the parameter.
To get it, it needs to obtain relation between u_1 and $\Psi^{\mathrm{I}}_{\pm}$:

$$(u_1, u_5) = (p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-})A_4$$
$$:= (p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-})\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Let us recall that we take the branch of $\sqrt{Q_0}$ as $\sqrt{Q_0} \sim \frac{\gamma}{2x}$ near the origin.

- **WKB solutions normalized at the origin:**

$$\psi^{(0)}_{\pm} := \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\pm \int_0^x (S_{\mathrm{odd}} - \frac{\gamma\eta + \gamma_0 - 1}{2x})dx \pm \int_1^x \frac{\gamma\eta + \gamma_0 - 1}{2x}\, dx\Big).$$

- **Borel summability**
  $\psi_{\pm}^{(0)}$ are Borel summable on each Stokes region. (Koike-Schäfke).
$\Psi_{\pm}^{(0),\mathrm{I}}(x,\eta)$ are the Borel sums of $\psi_{\pm}^{(0)}(x,\eta)$ in I.

If the WKB solution ψ_{+} is recessive solution, $\psi_{+}^{(0)}$ is recessive solution, too.

$$\text{When } |x| \to 0, \ \mathrm{Re} \int_1^x \frac{\gamma}{2x} dx \to -\infty.$$

- behavior of $\psi_{+}^{(0)}$ near the origin:

Since $S_{\mathrm{odd}} = \frac{\rho\eta}{x} + O(x)$,

$$\psi_{+}^{(0)} = \frac{x^{\rho\eta}}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\int_0^x (S_{\mathrm{odd}} - \frac{\rho\eta}{x}) dx \right) = \frac{1}{\sqrt{\rho\eta}} x^{\frac{1}{2}+\rho\eta}(1 + O(x)),$$

where

$$\rho\eta = \mathrm{Res}_{x=0} S_{\mathrm{odd}} = \frac{c-1}{2}. \tag{4}$$

- We set $\varphi_{+} := x^{-\frac{1}{2}-\rho\eta}\psi_{+}^{(0)} = x^{-c/2}\psi_{+}^{(0)}$.

Lemma 3 (Aoki-Takahashi-T [ATT])

(1) The formal series φ_+ for η is a Borel summable in a neighborhood of the origin and the Borel sum Φ_+ of φ_+ are holomorphic there. Here Φ_+ denote the Borel sum of φ_+.
(2) $\Phi_+(0,\eta) = \frac{1}{\sqrt{\rho\eta}}$. ($\mathrm{Res}_{x=0} S_{\mathrm{odd}} = \frac{c-1}{2} =: \rho\eta$.)

Taking the Borel sums of $\varphi_+ = x^{-c/2}\psi_+^{(0)}$, we have

$$\Phi_+(x,\eta) = x^{-c/2}\Psi_+^{(0),\mathrm{I}}(x,\eta).$$

By using the Lemma 3,

$$w := x^{-c/2}(1-x)^{-(a+b-c+1)/2}\Psi_+^{(0),\mathrm{I}}$$

are a holomorphic solution near the origin to the Gauss HGDE (1). The formal series φ_+ forms the one-dimensional subspaces of the solution spaces. There is a constant C_4 such that

$${}_2F_1(a,b,c;x) = C_4 x^{-c/2}(1-x)^{-(a+b-c+1)/2}\Psi_+^{(0),\mathrm{I}}.$$

Putting $x = 0$, we have

$$1 = C_4\sqrt{\frac{2}{c-1}}.$$

If $(\alpha, \beta, \gamma) \in \omega_4$, we have the following relation in a neighborhood of the origin:

$$u_1 = \sqrt{\frac{c-1}{2}} x^{-c/2} (1-x)^{-(a+b-c+1)/2} \Psi_+^{(0),\mathrm{I}}(x, \eta).$$

To obtain the asymptotic expansions of u_5, we use the relation between u_1 and $\Psi_+^{\mathrm{I}}(x, \eta)$ and the connection formula of Voros. Next we relate $\Psi_+^{\mathrm{I}}(x, \eta)$ to $\Psi_+^{(0),\mathrm{I}}(x, \eta)$.

The WKB solutions normalized at the turning point a_0:

$$\psi_\pm := \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{a_0}^{x} S_{\mathrm{odd}} dx \right),$$

- **The WKB solutions normalized at the origin:**

$$\psi_\pm^{(0)} := \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\pm \int_0^x (S_{\mathrm{odd}} - \frac{c-1}{2x}) dx \pm \int_1^x \frac{c-1}{2x}\, dx \Big).$$

Formally we have

$$\psi_{\pm}^{(0)} = \exp(\pm \hat{V}_0)\, \psi_{\pm,0}$$

with

$$\hat{V}_0 = \int_0^{a_0} \left(S_{\mathrm{odd}} - \frac{c-1}{2x} \right) dx + \frac{1}{2}(c-1) \log \tau_0,$$

where the branch of logarithm is taken suitably. Let $\hat{C}_0$ denote a path starting from the origin, encircles a_0 in a counterclockwise manner and returns to the origin.
We decompose $\hat{V}_0$ as the sum of two parts:

$$\hat{V}_0 = \sum_{j=-1}^{\infty} \eta^{-j} \hat{V}_{0,j} = V_0 + \hat{V}_{0,\leq 0},$$

with

$$V_0 = \frac{1}{2} \int_{\hat{C}_0} S_{\mathrm{odd},>0}\, dx,$$

$$\hat{V}_{0,\leq 0} = \lim_{x \to 0} \left(\int_x^{a_0} S_{\mathrm{odd},\leq 0}\, dx + \frac{(c-1)}{2} \log x \right).$$

We call V_0 the Voros coefficient of the origin.

Explicit form of $\hat{V}_0$

Theorem 4(Aoki-Takahashi-T[ATT1])

If $(\alpha,\beta,\gamma)\in\omega_4$, we have explicit form of $\hat{V}_0 = V_0 + \hat{V}_{0,\leq 0}$:

$$V_0 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}} + \frac{B_n(\beta_0)}{\beta^{n-1}} + \frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}} + \frac{B(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}} - \frac{B_n(\gamma_0)+B_n(\gamma_0-1)}{\gamma^{n-1}}\right),$$

$$\hat{V}_{0,\leq 0} = \frac{1}{2}\left(\frac{1}{2}-a\right)\log(-\alpha) + \frac{1}{2}\left(\frac{1}{2}-b\right)\log\beta + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\gamma-\alpha) + \frac{1}{2}\left(\frac{1}{2}-c+b\right)\log(\beta-\gamma) + (c-1)\log\gamma.$$

Here $B_n(x)$ $(n=0,1,2,\dots)$ denote the Bernoulli polynomials:

$$\frac{te^{xt}}{e^t-1} = \sum\frac{B_n(x)}{n!}t^n.$$

Borel sums of $\hat{V}_0$

$\hat{V}_0^4$: The Borel sum of $\hat{V}_0$ in ω_4.

Theorem 5(Aoki-Takahashi-T[ATT1])

If $(\alpha, \beta, \gamma) \in \omega_4$, the Borel sum $\hat{V}_0^4$ of $\hat{V}_0$ has the following form:

$$\hat{V}_0^4 = \frac{1}{2} \log \frac{\Gamma(1-a)\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)e^{(2a-c)\pi i}}{2\pi\Gamma(b)\Gamma(c-a)},$$

where $a = \alpha_0 + \alpha\eta, b = \beta_0 + \beta\eta, c = \gamma_0 + \gamma\eta$ and Γ is a Gamma function.

Taking the Borel sum of $\psi_\pm^{(0)} = \exp(\pm\hat{V}_0)\,\psi_{\pm,0}$ for $(\alpha, \beta, \gamma) \in \omega_4$, we have

$$\Psi_\pm^{(0),\mathrm{I}} = \exp(\pm\hat{V}_0^4)\,\Psi_+^{\mathrm{I}}.$$

If $(\alpha, \beta, \gamma) \in \omega_4$,

$$u_1 = \sqrt{\frac{\gamma_0 + \gamma\eta}{2}} \exp \hat{V}_0^4\, \Psi_+^{\mathrm{I}}.$$

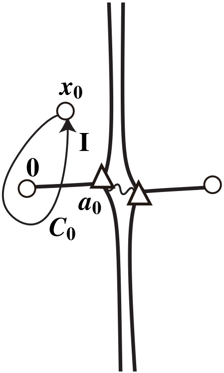
Hence we obtain the asymptotic expansion of u_1 as $\eta \to +\infty$:

$$u_1 \sim \sqrt{\frac{\gamma_0 + \gamma\eta}{2}} \exp \hat{V}_0^4 \psi_+.$$

Finally, we consider the following relation:

$$(u_1, u_5) = (p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-}) \begin{pmatrix} \sqrt{\gamma_0 + \gamma\eta/2} \exp \hat{V}^4_0 & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

$(\alpha, \beta, \gamma) \in \omega_4$



Let C_0 denote a closed path starting from x_0, going around the origin once counterclockwise and returning to x_0.

$(u_1, u_5)_{C_0}$ (resp.$(p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-})_{C_0}$) designates the analytic continuation of (u_1, u_5) (resp. $(p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-})$) along C_0.

Taking an analytic continuation of both sides of $(u_1, u_5) = (p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-})A_4$ for C_0, we have

$$(u_1, u_5)_{C_0} = (p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-})_{C_0} A_4$$

with $A_4 = \begin{pmatrix} \sqrt{\gamma_0 + \gamma\eta/2} \exp \hat{V}^4_0 & a_{12} \\ 0 & a_{22} \end{pmatrix}.$

The monodromy matrix of u_k $(k = 1, 5)$ for C_0 has the following form:

$$(u_1, u_5)_{C_0} = (u_1, u_5) \begin{pmatrix} 1 & 0 \\ 0 & \exp(-2c\pi i) \end{pmatrix}. \qquad (5)$$

On the other hand, using the connection formula of Voros, we have the following theorem.

Theorem 5(T [Tan])

If $(\alpha, \beta, \gamma) \in \omega_4$ and $x_0 \to a_0$, we have the monodromy matrix M_0^4 of C_0 satisfying

$$(\Psi_+^{\mathrm{I}}, \Psi_-^{\mathrm{I}})_{C_0} = (\Psi_+^{\mathrm{I}}, \Psi_-^{\mathrm{I}}) M_0^4.$$

($a = \alpha_0 + \eta\alpha, b = \beta_0 + \eta\beta, c = \gamma_0 + \eta\gamma$) Here

$$M_0^4 = \begin{pmatrix} \exp(c\pi i) & -i\exp(-c\pi i) \\ 0 & \exp(-c\pi i) \end{pmatrix}.$$

We recall the following relation:

$$(u_1, u_5) = (p(x)\Psi_+^{\mathrm{I}}, p(x)\Psi_-^{\mathrm{I}})A_4. \quad \left(A_4 = \begin{pmatrix} \sqrt{\gamma_0 + \gamma\eta/2}\exp\hat{V}_0^4 & a_{12} \\ 0 & a_{22} \end{pmatrix}.\right) \tag{6}$$

Taking an analytic continuation of both sides of it for C_0, we obtain

$$(u_1, u_5)\begin{pmatrix} 1 & 0 \\ 0 & \exp(-2c\pi i) \end{pmatrix} = (p(x)\Psi_+^{\mathrm{I}}, p(x)\Psi_-^{\mathrm{I}})\begin{pmatrix} \exp(-c\pi i) & 0 \\ 0 & \exp(-c\pi i) \end{pmatrix} M_0^4 A_4.$$

We rewrite it, we have

$$(u_1, u_5) = (p(x)\Psi^{\mathrm{I}}_{+}, p(x)\Psi^{\mathrm{I}}_{-}) \begin{pmatrix} \sqrt{\gamma_0 + \gamma\eta/2}\exp \hat{V}^4_0 & \exp(-2c\pi i)a_{21} - ia_{22} \\ 0 & a_{22} \end{pmatrix}. \tag{7}$$

Combining (6) **and** (7)**, we obtain**

$$a_{12} = -i\frac{\exp(2c\pi i)}{1 - \exp(2c\pi i)}a_{22}.$$

⋄ **Decision of constant a_{22}:** $\quad s(x) := \int_{a_0}^{x} S_{-1}dx.$

Lemma 6 (Aoki-Takahashi-T [ATT1])

If we take $d > 0$ suitably, we have the following form:

$$(\alpha, \beta, \gamma) \in \omega_4 : \lim_{\substack{x \to 0, x \in \mathrm{I} \\ |\mathrm{Im}\, s(x)| = d}} x^{c-1} p(x)\Psi^{\mathrm{I}}_{-} = \sqrt{\frac{2}{c-1}}\exp \hat{V}^4_0.$$

Here $p(x) = x^{-c/2}(1-x)^{-(a+b-c+1)/2}$.

Eliminating Ψ^{I}_{+} of

$$\begin{cases} u_1 = a_{11} p(x) \Psi^{\mathrm{I}}_{+}, \\ u_5 = a_{12}\, p(x) \Psi^{\mathrm{I}}_{+} + a_{22}\, p(x) \Psi^{\mathrm{I}}_{-}, \end{cases}$$

we obtain

$$x^{c-1} a_{22}\, p(x) \Psi^{\mathrm{I}}_{-} = x^{c-1} u_5 - a_{12}\, a_{11}{}^{-1} x^{c-1} u_1.$$

Putting $x = 0$, we have

$$a_{22} \sqrt{\frac{2}{c-1}}\, \exp \hat{V}^4_0 = 1$$

($u_1 = {}_2F_1(a, b, c; x)$ and $u_5 = x^{1-c}{}_2F_1(a+1-c, b+1-c, 2-c; x)$.)

We obtain

$$u_5 = \frac{i}{\exp(2\pi i c) - 1} \sqrt{\frac{c-1}{2}} \exp(-\hat{V}^4_0)\, p(x) \Psi^{\mathrm{I}}_{+} + \sqrt{\frac{c-1}{2}} \exp(-\hat{V}^4_0)\, p(x) \Psi^{\mathrm{I}}_{-}.$$

Then we have

$$u_5 \sim \frac{i}{\exp(2\pi i c) - 1} \sqrt{\frac{c-1}{2}} \exp(-\hat{V}^4_0)\, p(x) \psi_{+} + \sqrt{\frac{c-1}{2}} \exp(-\hat{V}^4_0)\, p(x) \psi_{-}.$$

Future work

• We want to obtain the asymptotic expansion of u_1 in the case where previous studies.
→ Using the result of my talk, we have the result for the case where $a = 1/2 + \eta\alpha, b = 1/2 + \eta\beta, c = 1$ needed to get the above result.
• It is necessary to prove whether the WKB solution is Borel summable for the case of the previous results' case.

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Thenk you for your attention