Complex powers of the wave operator
on asymptotically Minkowski spaces

joint work with Nguyen Viet Dang (Sorbonne Université)

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Introduction

Consider a Lorentzian manifold (M, g).

> The metric has signature $(+, -, \dots, -)$.
> For instance Minkowski space: $\mathbb{R}^{1+d}$, $g_0 = dt^2 - dy_1^2 - \cdots - dy_{n-1}^2$.

The Lorentzian Laplace–Beltrami operator or wave operator:

$$\Box_g = \sum_{i,j=0}^{n-1} |g(x)|^{-\frac{1}{2}} \partial_{x^i} |g(x)|^{\frac{1}{2}} g^{ij}(x) \partial_{x^j}$$

On Minkowski space, $\Box_g = \partial_t^2 - (\partial_{y_1}^2 + \cdots + \partial_{y_{n-1}}^2)$.

> $\Box_g$ (+ non-linearity) has rich theory of solving Cauchy problem, asymptotic analysis of solutions, propagation of singularities, etc.

Relatively recently: global theory of $\Box_g$ (Fredholm property, Hilbert space invertibility) Vasy '13 et al.. Techniques of microlocal and asymptotic analysis in relation with classical dynamics and geometry.

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> $\Box_g$ (+ non-linearity) has rich theory of solving Cauchy problem, asymptotic analysis of solutions, propagation of singularities, etc.

As opposed to $\triangle_g$ on Riemannian manifold, $\Box_g$ is non-elliptic.

? How is global $\Box_g$ related to geometry of (M, g)?

Spectral zeta function

(M, g) compact Riemannian $\implies \triangle_g$ has discrete spectrum.

Recall Riemann zeta $\zeta(\alpha) = \sum_{\lambda=1}^{\infty} \lambda^{-\alpha}$, then spectral zeta:

$$\mathbb{C} \ni \alpha \mapsto \zeta_{\triangle}(\alpha) = \sum_{\lambda \in \mathrm{sp}(\triangle_g) \setminus \{0\}} \lambda^{-\alpha}.$$

Theorem (Minakshisundaram–Pleijel, Seeley)

The function $\zeta_{\triangle}(\alpha) = \mathrm{Tr}_{L^2}\left(\triangle_g^{-\alpha}\right)$ *is holomorphic on* $\mathrm{Re}\,\alpha > \frac{n}{2}$*, with meromorphic continuation to* $\alpha \in \mathbb{C}$ *and poles at* $\{\frac{n}{2}, \frac{n}{2} - 1, \ldots, 1\}$*.*

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+local version with densities:

$\alpha \mapsto \triangle_g^{-\alpha}(x,x)$ holomorphic on $\mathrm{Re}\,\alpha > \frac{n}{2}$, with meromorphic continuation to $\alpha \in \mathbb{C}$ and poles at $\{\frac{n}{2}, \frac{n}{2}-1, \dots, 1\}$, smooth in $x \in M$.

> Here $\triangle_g^{-\alpha}(x,x')$ is the Schwartz kernel of $\triangle^{-\alpha}$, so
>
> $$\mathrm{Tr}_{L^2}\left(\triangle_g^{-\alpha}\right) = \int_M \triangle_g^{-\alpha}(x,x)dx$$

The spectral action principle

The heat kernel expansion (small t expansion of $e^{-t\triangle_g}(x,x)$) relates $\triangle_g$ with invariants, in particular scalar curvature $R_g(x)$.

Theorem (well-known from elliptic theory)

When dim$(M) = n \geqslant 4$*,*

$$\operatorname*{res}_{\alpha=\frac{n}{2}-1} \mathrm{Tr}_{L^2}\left(\triangle_g^{-\alpha}\right) = \frac{\int_M R_g(x)}{6(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)}.$$

Local version for diagonal value $x = x'$ of Schwartz kernel $\triangle_g^{-\alpha}(x,x')$:

$$\operatorname*{res}_{\alpha=\frac{n}{2}-1} \triangle_g^{-\alpha}(x,x) = \frac{R_g(x)}{6(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)}.$$

- This is a "spectral action" for Euclidean gravity: $\delta_g R_g = 0$ is equivalent to Einstein equations.
- Poles are geometric $\Rightarrow$ locality of counterterms in zeta function regularisation in QFT Hawking '77

Theorem (well-known from elliptic theory / semi-classical analysis)

For any Schwartz function f,

$$f\big(\triangle_g/\lambda^2\big)(x,x) = \sum_{j=0}^{N} \lambda^{n-2j} C_j(f)\, a_j(x) + \mathcal{O}(\lambda^{n-2N-1}),$$

where $a_j(x)$ are the heat kernel coefficients.

- There is a vectorial version for Dirac operators $f\big(\frac{D\!\!\!/^2}{\lambda^2}\big)$.
- Twisting the bundle yields Standard model Lagrangian
  Chamseddine–Connes ’97

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But no direct physical meaning unless (M,g) Lorentzian...

Fundamental difficulties: Lorentzian $\Box_g$ **not elliptic**, not bounded from below. There is no Lorentzian heat kernel.

So what about $\Box_g$?

For (M, g) Lorentzian, Δ_g becomes $\Box_g$. Two hints:

1. The **local geometric quantities** (e.g. $R_g(x)$) still make sense.
   - Lorentzian version of local heat kernel coefficients $a_j(x)$ by solving analogous transport equations
2. Recent results show **essential self-adjointness** of $\Box_g$:
   - Static spacetimes (e.g. $\partial_t^2 - \Delta_h$ with time-independent coefficients): Dereziński-Siemssen '18
   - For perturbations of Minkowski space (and more general non-trapping Lorentzian scattering spaces): Vasy '20 (short-range), Nakamura–Taira '20 (long-range) (related results: Gérard–Wrochna '19–'20, Kamiński '19, Dereziński-Siemssen '19, Colin de Verdière–Le Bihan '20, Taira '20)
   - Asymptotically static spacetimes Nakamura–Taira '22

$\Rightarrow f(\Box_g)$ well-defined!

But is there any relationship between **1.** and **2.** like in elliptic case?

I. Main results

Main theorem

Assume (M,g) is a (short-range) perturbation of Minkowski space (or more general non-trapping Lorentzian scattering space, see later), of even dimension n.

Theorem (Dang, Wrochna)

For $\varepsilon > 0$, the Schwartz kernel of $(\Box_g - i\varepsilon)^{-\alpha}$ has for Re $\alpha > \frac{n}{2}$ *a well-defined on-diagonal restriction $(\Box_g - i\varepsilon)^{-\alpha}(x,x)$, which extends as a meromorphic function of $\alpha \in \mathbb{C}$ with poles at $\{\frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \dots, 1\}$. Furthermore,*

$$\lim_{\varepsilon\to 0^+} \operatorname*{res}_{\alpha=\frac{n}{2}-1} \left(\Box_g - i\varepsilon\right)^{-\alpha}(x,x) = \frac{R_g(x)}{i6(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)},$$

where $R_g(x)$ is the scalar curvature at $x \in M$.

— **Spectral action for gravity**. Proof directly in Lorentzian signature. Perturbations of Minkowski included (no symmetries assumed).

— The $\varepsilon \to 0^+$ avoids low-frequency problems and responsible for relationship with Feynman propagator.

Main theorem 2

Assume (M, g) is a (short-range) perturbation of Minkowski space (or more general non-trapping Lorentzian scattering space, see later), of even dimension n.

Theorem (Dang, Wrochna)

For any Schwartz f with Fourier transform in $]0, +\infty[$,

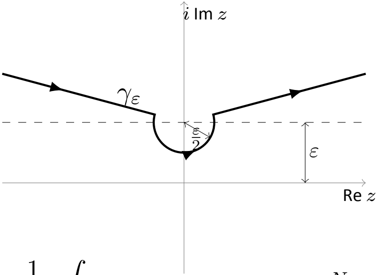
$$f\big((\Box_g + i\varepsilon)/\lambda^2\big)(x,x) = \sum_{j=0}^{N} \lambda^{n-2j} C_j(f)\, a_j(x) + \mathcal{O}(\varepsilon, \lambda^{n-2N-1}),$$

where $a_j(x)$ are Hadamard coefficients.

II. From resolvent to geometric invariants

General plan of proof

1) Let $P = \Box_g$ on Lorentzian (M, g). If resolvent exists, $(P - i\varepsilon)^{-\alpha}$ obtained as contour integral of $(P - z)^{-1}$. For $\alpha = N + \mu > 0$:



$$(P - i\varepsilon)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (z - i\varepsilon)^{-\mu} (P - i\varepsilon)^{-N} (P - z)^{-1} dz, \quad .$$

2) Construct a Hadamard parametrix $H_N(z)$ and show it approximates the resolvent uniformly in z.

3) Deduce regularity properties, compute poles and get curvature R from contour integrals of $H_N(z)$.

Construction of Hadamard parametrix $H_N(z)$:

Let $\mathsf{F}_\alpha(z, |.|_g)$ be locally given by

$$F_\alpha(z,x) = \frac{1}{\Gamma(\alpha+1)(2\pi)^n} \int e^{i\langle x,\xi\rangle} \left(|\xi|_{g_0}^2 - i0 - z\right)^{-\alpha-1} d^n\xi$$

(in normal coordinates). Candidate for parametrix of order N:

$$H_N(z,.) = \sum_{k=0}^{N} u_k \mathsf{F}_k(z, |.|_g) \in \mathcal{D}'(\mathcal{U}).$$

solved modulo errors by transport equations thanks to

$$(P-z)\left(u\mathsf{F}_\alpha\right) = \alpha u \mathsf{F}_{\alpha-1} + (Pu)\mathsf{F}_\alpha + (hu + 2\rho u)\frac{\mathsf{F}_{\alpha-1}}{2}$$

for all $u \in C^\infty(M)$, where $h(x) = b^j(x) g_{0,jk} x^k$ and $\rho = x^k \partial_{x^k}$.

Hölder–Zygmund and microlocal estimates for $F_\alpha(z, |.|_\eta)$

- competition between regularity in x and decay in z
- use of Hadamard parametrix wide-spread in QFT; also analytic continuation of eigenfunctions Zelditch '18 and Lorentzian local index theory Bär–Strohmaier '20. We followed mostly Hörmander vol. 3 and Sogge '14
- z-dependent Hadamard in compact Riemannian setting used by Sogge '88, Dos Santos Ferreira–Kenig–Salo '14, Bourgain–Shao–Sogge–Yao '15

Compute poles and get curvature:

Now $(P - i\varepsilon)^{-\alpha}(x, x)$ expressed by contour integrals of $\mathbf{F}_\beta(z, .)$.

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z - i\varepsilon)^{-\alpha}\mathbf{F}_k(z, .)dz = \frac{(-1)^k\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)}\mathbf{F}_{k+\alpha-1}(i\varepsilon, .)$$

scalar curvature in normal coordinates comes from

$$P = \partial_{x^k} g^{kj}(x)\partial_{x^j} + g^{jk}(x)(\partial_{x^j}\log|g(x)|^{\frac{1}{2}})\partial_{x^k},$$

transport equation $u_1(0) = -Pu_0(0) = -P(|g(0)|^{\frac{1}{4}}\,|g(x)|^{-\frac{1}{4}})|_{x=0}$
and $g_{ij}(x) = g_{0,ij} + \frac{1}{3}R_{ikjl}x^k x^l + \mathcal{O}(|x|^3)$.

Hadamard parametrix H_N approximates $(P-z)^{-1}$?

$$(P-z)\left(\sum_{k=0}^{N} u_k \mathbf{F}_k(z,.)\chi\right) = |g|^{-\frac{1}{2}}\delta_\Delta + (Pu_N)\mathbf{F}_N(z,.)\chi + r_N(z),$$

where Pu_N highly regular, and r_N singular (but 0 near diagonal). Applying $(P-z)^{-1}$ well-defined and yields good errors if $(P-z)^{-1}$ is shown to have special structure of *singularities* and mapping properties uniformly in z.

> Think of the distribution $(x-i0)^{-1}$ on $\mathbb{R}$: it is singular at $x=0$, but has good multiplicative properties like $(x-i0)^{-1}(x-i0)^{-1}=(x-i0)^{-2}$.

Here, "controlling singularities" means showing existence of $B_1, B_2 \in \Psi^0(M)$, as elliptic as possible s.t.

$$B_1(P-z)^{-1}B_2^* : L^2(M) \to C^\infty(M)$$

with seminorms $O(1+|z|)^{-\frac{1}{2}}$. In our case, possible except if B_1 forward connected with B_2 (in other words, $(P-z)^{-1}$ has Feynman wavefront set).

III. Analysis of $(P - z)^{-1}$

Suppose $P = \partial_t^2 - \triangle$, Im $z > 0$. Retarded propagator of $P - z$:

$$\theta(t-s)\frac{e^{i(t-s)\sqrt{-\triangle-z}} - e^{-i(t-s)\sqrt{-\triangle-z}}}{2i\sqrt{-\triangle - z}}$$

Looks like no chance of $\|(P-z)^{-1}\| \leqslant |\text{Im}\, z|^{-1}$. But:

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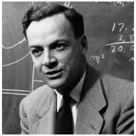

"Every particle in Nature has an amplitude to move backwards in time, and therefore has an anti-particle."

– Richard Feynman

$$\big((P-z)^{-1}u\big)(t,.) = -\frac{1}{2}\int \frac{e^{-i|t-s|\sqrt{-\triangle - z}}}{\sqrt{-\triangle - z}} u(s,.)ds. \qquad (1)$$

The boundary value $(P - i0)^{-1}$ is the Feynman propagator.

But for general P with t-dependent coefficients, nothing like (1) exists...

- Start with (1) at infinity, then propagate!

Suppose $P = \partial_t^2 - \triangle$, Im $z > 0$. Retarded propagator of $P - z$:

$$\theta(t-s)\frac{e^{i(t-s)\sqrt{-\triangle-z}} - e^{-i(t-s)\sqrt{-\triangle-z}}}{2i\sqrt{-\triangle-z}}$$

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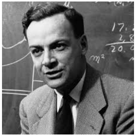

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The boundary value $(P - i0)^{-1}$ is the Feynman propagator.

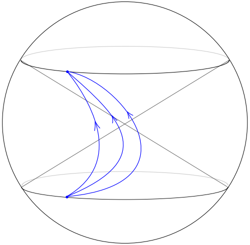
- Use radial estimates due to Melrose '94 and Vasy '13-'19 (or assume g is a compactly supported perturbation of Minkowski metric g_0) + propagation estimates Hörmander '71

Lorentzian scattering spaces

Example: Minkowski metric $g_0 = dx_0^2 - (dx_1^2 + \cdots + dx_{n-1}^2)$ on $\mathbb{R}^n$ extends to radial compactification $\overline{\mathbb{R}}^n$ defined using boundary-defining function $\rho = (x_0^2 + x_1^2 + \cdots + x_n^2)^{-\frac{1}{2}}$. Regularity w.r.t. $\rho^2 \partial_\rho = -\partial_r$

Definition: Lorentzian sc-metrics are C^∞ sections of ${}^{\mathrm{sc}}T^*M \otimes_{\mathrm{s}} {}^{\mathrm{sc}}T^*M$, where ${}^{\mathrm{sc}}T^*M$ generated by $\rho^{-2}d\rho$, $\rho^{-1}dy_1, \ldots \rho^{-1}dy_{n-1}$.

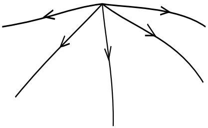
Null geodesics lift to null bicharacteristics on ${}^{\mathrm{sc}}T^*M$ (rescaled and extended at ∂M appropriately)



Definition:
(M, g) non-trapping Lorentzian sc-space
if there are sinks/sources $L_\pm$ above ∂M, and
null bicharacteristics flow from and to L_- and L_+.

Includes small **perturbations of Minkowski space** *and* **asymptotically Minkowski** *spaces.*

From null bicharacteristic flow to global estimates



dynamics of null bicharacteristics in $\overline{{}^{\mathrm{sc}}T^*M}$

$\Downarrow$

classical quantities increasing along flow

$\Downarrow$

pos. commutator estimates in $\Psi_{\mathrm{sc}}^{m,\ell}$-calculus

$\Downarrow$

1. Deduce Fredholm property and invertibility of $P - z$
2. Deduce singularities of $(P - z)^{-1}(x, x')$

problem of computing $\mathrm{WF}\big((P-z)^{-1}\big)$ analogous to Dyatlov–Zworski ’16, but our strategy closer to Vasy–Wrochna ’18 + z-dependent calculus of Shubin ’01, parametrix similar to Gérard–Wrochna ’19

work in progress with N.V. Dang and A. Vasy: $\mathrm{WF}\big((P-z)^{-1}\big)$ directly from (improved) estimates, also for non-selfadjoint generalisations of the problem

IV. Summary

To sum up...

We have shown relationship of Lorentzian spectral zeta function density $\zeta_{g,\varepsilon}$ with space-time geometry.

$\Rightarrow$ *(Lorentzian!) Gravity can be derived from a spectral action*.

- We also get the theorem for ultra-static spacetimes and compactly supported pertubations. One can conjecture extensions to asymptotically static spacetimes (and beyond, especially if weakening essential self-adjointness).
- Relationships with QFT on curved spacetimes : $(\Box_g - i\varepsilon)^{-\alpha}$ useful in zeta renormalization.

  Remarks: $(\Box_g - z)^{-1}$ is *not* a retarded or advanced propagator, but a Feynman propagator: turns out to have better properties in non-linear problems Gell-Redman–Haber–Vasy '16.
- Is there a version (even of the Hadamard parametrix) for anti-de Sitter spacetimes?

Thank you for your attention!

IV. Appendix

Positive commutator estimates

Toy model: $P = P^*$ bounded, and $\exists$ bounded A and D s.t.:

$$[P, iA] \geqslant (\mathbf{1} + D^2)^s. \qquad (*)$$

Undo the commutator:

$$\begin{aligned}\frac{1}{2}\langle [P, iA]u, u\rangle &= \frac{\langle APu, u\rangle - \langle PAu, u\rangle}{2i} \\ &= \frac{\langle Pu, Au\rangle - \langle Au, Pu\rangle}{2i} \leqslant |\langle Pu, Au\rangle|,\end{aligned}$$

By Cauchy–Schwarz,

$$|\langle Pu, Au\rangle| \leqslant C\|(\mathbf{1} + D^2)^{-s/2} Pu\| \|(\mathbf{1} + D^2)^{s/2} u\| =: C\|Pu\|_{-s}\|u\|_s.$$

In combination with (*):

$$\|u\|_s^2 \leqslant C\|Pu\|_{-s}\|u\|_s,$$

hence invertibility statement $\|u\|_s \leqslant C\|Pu\|_{-s}$.

Positive commutator estimates

The existence of suitable A s.t.

$$[P, iA] \geqslant (\mathbf{1} + D^2)^s.$$

is extremely rare. But we can expect to prove it “somewhere in phase space”.

- If $P \in \Psi^s(M)$ and $A \in \Psi^\ell(M)$ then $[P, iA] \in \Psi^{s+\ell-1}(M)$ and

  $$\sigma_{\mathsf{pr}}([P, iA]) = \{p, a\} \text{ mod } S^{s+\ell-2}(M).$$

  The flow of $\{p, \cdot\}$ in $\{p = 0\}$ is the classical Hamilton flow, or **bicharacteristic flow** (note that in $\{p \neq 0\}$ elliptic theory applies).

- non-compact settings require weighted Sobolev spaces: extra weight $(\mathbf{1} + |x|^2)^\ell$ ($\Psi_{\mathsf{sc}}^{m,\ell}(M)$ calculus)

- non-selfadjointness can be serious trouble (if we know nothing of $P - P^*$), or valuable help (for instance $P - i\varepsilon$ with $\varepsilon > 0$)

Dirac operators

The Lorentzian Dirac operator $\not{D}$ satisfies $\not{D}^2 = \Box_g +$ l.o.t. in vector bundle sense. It is formally self-adjoint w.r.t. the canonical indefinite inner product, but (in general) not for an honest scalar product. However, on Lorentzian scattering spaces, $P := \not{D}^2$ satisfies

$$P^* - P \in \Psi_{\mathrm{sc}}^{1,-1-\delta}(M)$$

for instance for the scalar product $\langle \cdot, \gamma(n) \cdot \rangle_{L^2(M;SM)}$ used in quantization

- work in progress (with N.V. Dang & A. Vasy): $P = \not{D}^2$ on non-trapping Lorentzian scattering space (M, g) as closed operator.

> **Conjecture**
>
> $\not{D}^2$ is a closed operator, and:
>
> $$\mathrm{sp}(\not{D}^2) \subset \mathbb{R} \cup \{\text{some isolated poles in } |\mathrm{Im}\, z| \leqslant R\}$$

This uses stronger resolvent estimates using a resolved $\Psi_{\mathrm{sc}}^{m,\ell}$-calculus obtained from blowing up the corner of ${}^{\mathrm{sc}}\overline{T^*}M$.

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The techniques give a fully microlocal implementation of subelliptic estimate of Taira '21:

$u \in H_{\mathrm{sc}}^{m+\frac{1}{2},\ell-\frac{1}{2}}(M)$, $(P - z)u \in H_{\mathrm{sc}}^{m,\ell}(M) \Rightarrow u \in H_{\mathrm{sc}}^{m,\ell}(M)$.

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Remark: No role played by indefinite $\langle \cdot, \cdot \rangle_{L^2(M;SM)}$