A depth-dependent stability estimate in an iterative method for solving a Cauchy problem for the Laplace equation

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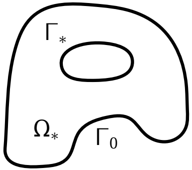
Ω_*: bounded domain in $\mathbb{R}^2$

$\partial\Omega_* = \Gamma_0 \cup \Gamma_*$ (Γ_0, Γ_*: closed, disjoint).

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_*, \\ u = \varphi & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = \psi & \text{on } \Gamma_0. \end{cases} \tag{1.1}$$



ν: outward unit vector normal of $\partial\Omega_*$.

In general, for given φ and ψ,
there does not always exist a solution u to (1.1) in Ω_*.

We now assume that there exists the solution u to (1.1) in Ω_*.
Then, we consider a stability of an approximate solution obtained by the iterative procedure for φ and ψ.

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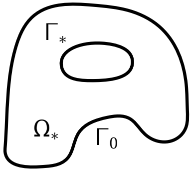


ν: outward unit vector normal of $\partial\Omega_*$.

Related works

Bastay-Kozlov-Turesson (2001)

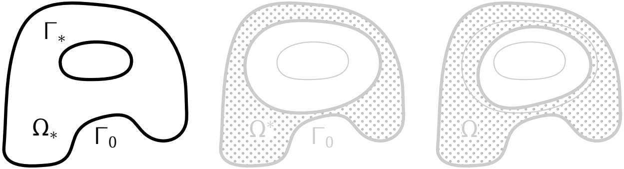
- one-step stationary iterative method
- minimal error method

elliptic: Johansson (2004, 2006) ...

parabolic: Chapko-Johansson-Vavrychuk (2013), Johansson (2006)

In the Bastay-Kozlov-Turesson iteration, we construct a sequence of approximate solutions to (1.1) by solving boundary value problems repeatedly.

- If we would like to construct the solution to (1.1) in Ω_*, we need to solve the corresponding boundary value problems in the whole of domain Ω_*.
- If we would like to construct the solution to (1.1) in $\Omega^*(\subset \Omega_*)$ near Γ_0, we can also consider the boundary value problems only in Ω satisfying $\Omega^* \subset \Omega \subset \Omega_*$. Then, we expect that we can construct approximate solutions more stably when we choose smaller domain Ω.



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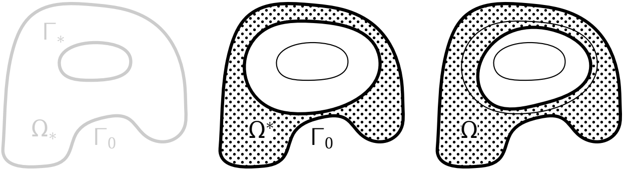

X, Y: Hilbert space. $K : X \to Y$: linear, compact, injective.

Problem

For given $y \in Y$, we find $x \in X$ satisfying $Kx = y$.

Let $a > 0$ and $Kx = y$. K^*: adjoint operator to K. I: identity operator.

Landweber iteration (z: initial data)

$$x^{(0)} = z, \quad x^{(m)} = (I - aK^*K)x^{(m-1)} + aK^*y. \tag{2.1}$$

By solving the recurrence relation (2.1), we obtain

$$x^{(m)} = \widetilde{R_m}y, \text{ where } \widetilde{R_m}y = (I - aK^*K)^m z + a\sum_{k=0}^{m-1}(I - aK^*K)^k K^*y. \tag{2.2}$$

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$\|K\|_{\mathcal{L}}$ denotes the operator norm.

Theorem 1

Let $0 < a < \frac{2}{\|K\|_{\mathcal{L}}^2}$ *and* $z \in X$*. If* $Kx^0 = y^0$*, then*

$$\|\widetilde{R}_m y^0 - x^0\|_X \to 0 \quad \text{as } m \to \infty.$$

Theorem 2

Let $0 < a < \frac{2}{\|K\|_{\mathcal{L}}^2}$, $\delta > 0$ *and* $z \in X$*. Moreover, we choose* $m(\delta) > 0$ *such that*

$$\lim_{\delta \to +0} \delta^2 m(\delta) = 0, \quad \lim_{\delta \to +0} m(\delta) = \infty.$$

For y^δ *with* $\|y^\delta - y^0\|_Y \leq \delta$*, if* $Kx^0 = y^0$*, then*

$$\lim_{\delta \to +0} \|\widetilde{R}_{m(\delta)} y^\delta - x^0\|_X = 0.$$

Theorem 2

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$$\underline{\lim_{\delta \to +0} \delta^2 m(\delta) = 0} \ , \quad \underline{\lim_{\delta \to +0} m(\delta) = \infty} \ .$$

For y^δ *with* $\|y^\delta - y^0\|_Y \leq \delta$, *if* $Kx^0 = y^0$, *then*

$$\lim_{\delta \to +0} \|\widetilde{R}_{m(\delta)} y^\delta - x^0\|_X = 0.$$

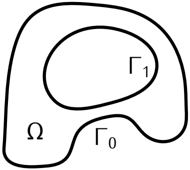
$$\|\widetilde{R}_m y^\delta - x^0\|_X \leq \|\widetilde{R}_m y^\delta - \widetilde{R}_m y^0\|_X + \|\widetilde{R}_m y^0 - x^0\|_X$$

error from:	observation $y^\delta - y^0$	iteration
$m \to \infty$:	$\to \infty$	$\to 0$

Thus, we have to choose $\underline{m \text{ depending on } \delta}$.

At each step, we solve the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \eta & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial \nu} = \psi & \text{on } \Gamma_0, \end{cases} \tag{3.1}$$



and the adjoint problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial v}{\partial \nu} = \zeta & \text{on } \Gamma_0. \end{cases} \tag{3.2}$$

Problems (3.1) and (3.2) have unique solutions, respectively.

- Choose an arbitrary function $\eta^{(0)} \in L^2(\Gamma_1)$.
- The first approximation $u^{(0)}$ to the solution u is obtained by solving the problem (3.1) with $\eta = \eta^{(0)}$ on Γ_1.

$$\begin{cases} \Delta u^{(0)} = 0 & \text{in } \Omega, \\ u^{(0)} = \eta^{(0)} & \text{on } \Gamma_1, \\ \dfrac{\partial u^{(0)}}{\partial \nu} = \psi & \text{on } \Gamma_0. \end{cases}$$

- Then, we find the auxiliary function $v^{(0)}$, which is given by the solution to the problem (3.2) with $\zeta = \zeta^{(0)}$, where $\zeta^{(0)} = u^{(0)} - \varphi$ on Γ_0.

$$\begin{cases} \Delta v^{(0)} = 0 & \text{in } \Omega, \\ v^{(0)} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial v^{(0)}}{\partial \nu} = u^{(0)} - \varphi & \text{on } \Gamma_0. \end{cases}$$

- When the solutions $u^{(\ell-1)}$ and $v^{(\ell-1)}$ have been constructed, the approximation $u^{(\ell)}$ is the solution to (3.1) with data $\eta = \eta^{(\ell)}$ on Γ_1, where

$$\eta^{(\ell)} = u^{(\ell-1)} + \gamma \frac{\partial v^{(\ell-1)}}{\partial \nu}$$

  and γ is a fixed positive number.

$$\begin{cases} \Delta u^{(\ell)} = 0 & \text{in } \Omega, \\ u^{(\ell)} = \eta^{(\ell)} & \text{on } \Gamma_1, \\ \dfrac{\partial u^{(\ell)}}{\partial \nu} = \psi & \text{on } \Gamma_0. \end{cases}$$

- The auxiliary function $v^{(\ell)}$ is the solution to (3.2) with data $\zeta = \zeta^{(\ell)}$, where $\zeta^{(\ell)} = u^{(\ell)} - \varphi$ on Γ_0.

$$\begin{cases} \Delta v^{(\ell)} = 0 & \text{in } \Omega, \\ v^{(\ell)} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial v^{(\ell)}}{\partial \nu} = u^{(\ell)} - \varphi & \text{on } \Gamma_0. \end{cases}$$

Definition 3

We introduce an operator $K : L^2(\Gamma_1) \to L^2(\Gamma_0)$ through

$$K\eta = z_1|_{\Gamma_0} \quad \text{for } \eta \in L^2(\Gamma_1),$$

where z_1 is the solution to (3.1) with $\psi = 0$.

Similarly, we define the operator $K_1 : L^2(\Gamma_0) \to L^2(\Gamma_0)$ by

$$K_1\psi = z_2|_{\Gamma_0} \quad \text{for } \psi \in L^2(\Gamma_0),$$

where z_2 is the solution to (3.1) with $\eta = 0$.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \eta & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial \nu} = \psi & \text{on } \Gamma_0, \end{cases} \tag{3.1}$$

$$K\eta = z_1\big|_{\Gamma_0}, \text{ where } z_1 \text{ is the solution to (3.1) with } \psi = 0,$$
$$K_1\psi = z_2\big|_{\Gamma_0}, \text{ where } z_2 \text{ is the solution to (3.1) with } \eta = 0.$$

- K: compact, injective. K_1: compact.
- Solving (1.1) is equivalent to solving the following equation

$$K\eta = \varphi - K_1\psi. \tag{3.3}$$

If η is a solution to (3.3)*, then the solution to* (3.1) *satisfies $u = \varphi$ on Γ_0 and thus solves* (1.1)*. Conversely, if u solves* (1.1)*, then $\eta = u|_{\Gamma_1}$ is a solution to* (3.3)*.*

- The adjoint $K^* : L^2(\Gamma_0) \to L^2(\Gamma_1)$ to the operator K is given by

$$K^*\zeta = -\left(\frac{\partial v}{\partial \nu}\right)\bigg|_{\Gamma_1} \quad \text{for } \zeta \in L^2(\Gamma_0),$$

where v is the solution to (3.2).

$$K\eta = \varphi - K_1\psi. \tag{3.3}$$

Moreover, from the algorithm, we have

$$\eta^{(\ell)} = (I - \gamma K^*K)\eta^{(\ell-1)} + \gamma K^*(\varphi - K_1\psi). \tag{3.4}$$

Since (3.4) is the Landweber iteration for (3.3), from Theorem 1, we have

$$\|\eta^{(\ell)} - \eta\|_{L^2(\Gamma_1)} \to 0 \quad \text{as } \ell \to \infty.$$

Furthermore, by the well-posedness of (3.1), we obtain

$$\|u^{(\ell)} - u\|_{L^2(\Omega)} \leq C'\|\eta^{(\ell)} - \eta\|_{L^2(\Gamma_1)} \to 0 \quad \text{as } \ell \to \infty.$$

Theorem 4 (Bastay-Kozlov-Turesson, 2001)

Let $u \in L^2(\Omega)$ *be the solution to* (1.1). *We suppose that* $0 < \gamma < \dfrac{2}{\|K\|^2_{L^2(\Gamma_1)\to L^2(\Gamma_0)}}$.

Let $u^{(\ell)}$ *be the* ℓ*-th approximate solution.*

Then, for every initial data function $\eta^{(0)} \in L^2(\Gamma_1)$,

$$\|u^{(\ell)} - u\|_{L^2(\Omega)} \to 0 \quad \textit{as } \ell \to \infty.$$

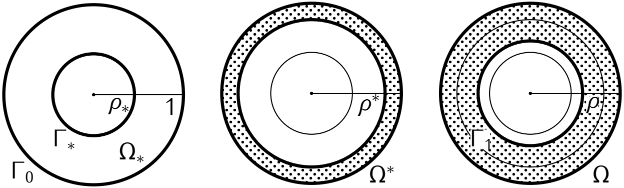
"We choose the smaller domain where we consider the boundary value problems, we construct them more stably."

In order to show this, we consider the following situation.

Let $0 < \rho_* \leq \rho \leq \rho^* < 1$.

$$\Omega_* = \{x \in \mathbb{R}^2 \mid \rho_* < |x| < 1\},\ \Omega^* = \{x \in \mathbb{R}^2 \mid \rho^* < |x| < 1\},\ \Omega = \{x \in \mathbb{R}^2 \mid \rho < |x| < 1\},$$

$$\Gamma_0 = \{x \in \mathbb{R}^2 \mid |x| = 1\},\ \Gamma_* = \{x \in \mathbb{R}^2 \mid |x| = \rho_*\},\ \Gamma_1 = \{x \in \mathbb{R}^2 \mid |x| = \rho\}.$$



Theorem 5 (I.)

$$(\sharp)\left(\begin{array}{l} \textit{Given positive numbers } M_0 \textit{ and } M. \\ \textit{We suppose that } \varphi \textit{ and } \psi \textit{ are real-valued functions} \\ \textit{with } \|\varphi\|^2_{L^2(\Gamma_0)} + \|\psi\|^2_{L^2(\Gamma_0)} \le M_0^2. \\ \textit{We assume that there exists the solution } u \\ \textit{to the Cauchy problem } (1.1) \textit{ in } \Omega_* \textit{ and its trace on } \Gamma_* \textit{ is in } L^2. \\ \textit{We suppose that } \|u\|_{L^2(\Gamma_*)} \le M. \end{array}\right)$$

Let $u^{(\ell)}$ be an approximate solution obtained by the iterative procedure with $\eta^{(0)} = 0$ and $\gamma = \rho$. Then, for $\ell \geq 2$ and $\rho \in [\rho_*, \rho^*]$, the following estimate holds:

$$\|u^{(\ell)} - u\|^2_{L^2(\Omega^*)} \le C \left(\frac{\log \ell}{\ell} \right)^{\min\left\{ \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1 \right\}}, \tag{4.1}$$

where a positive constant C depends only on M_0, M, ρ_* and ρ^*.

$$\|u^{(\ell)} - u\|^2_{L^2(\Omega^*)} \leq C \left(\frac{\log \ell}{\ell} \right)^{\min\left\{ \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1 \right\}}. \tag{4.1}$$

Remark 6

We have

$$\min\left\{ \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1 \right\} = \begin{cases} \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)} & \text{for } \rho_* \leq \rho < \frac{\rho_*}{\rho^*}, \\ 1 & \text{for } \frac{\rho_*}{\rho^*} \leq \rho \leq \rho^*. \end{cases}$$

There exists no ρ satisfying $\frac{\rho_*}{\rho^*} \leq \rho \leq \rho^*$ in the case where $\sqrt{\rho_*} > \rho^*$. Namely, in this case, we have

$$\|u^{(\ell)} - u\|^2_{L^2(\Omega^*)} \leq C \left(\frac{\log \ell}{\ell} \right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}}$$

for any ρ.

$$\|u^{(\ell)} - u\|^2_{L^2(\Omega^*)} \leq C \left(\frac{\log \ell}{\ell} \right)^{\min\left\{ \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1 \right\}}. \tag{4.1}$$

The larger ρ we choose, the larger the power $\dfrac{\log(\rho^*/\rho_*)}{\log(1/\rho)}$ on the right-hand side of (4.1) is. It means that we have the better stability as we choose ρ larger. Moreover, with regard to the optimality of (4.1), we have the following theorem.

Theorem 7 (I.)

Let $\varepsilon > 0$ be given. Under the assumptions in Theorem 5, there exists no positive constant $\widehat{C}$ depending only on M_0, M, ρ_* and ρ^* such that

$$\|u^{(\ell)} - u\|^2_{L^2(\Omega^*)} \leq \widehat{C} \left(\frac{\log \ell}{\ell} \right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)} + \varepsilon} \tag{4.2}$$

holds for any $\ell \geq 2$ and $\rho \in [\rho_*, \rho^*]$.

Since the Landweber iteration works with inexact data,
the Bastay-Kozlov-Turesson iteration also works.

We next consider the case where we know only approximations φ^δ and ψ^δ:

$$\|\varphi^\delta - \varphi\|_{L^2(\Gamma_0)} \leq \delta, \quad \|\psi^\delta - \psi\|_{L^2(\Gamma_0)} \leq \delta. \tag{4.2}$$

Let $u^{(\ell),\delta}$ be an approximation obtained by the iterative procedure with $\eta^{(0)} = 0$ and $\gamma = \rho$ for φ^δ and ψ^δ.

Since we have

$$\begin{aligned} \|u^{(\ell),\delta} - u\|_{L^2(\Omega^*)} &\leq \|u^{(\ell),\delta} - u^{(\ell)}\|_{L^2(\Omega^*)} + \|u^{(\ell)} - u\|_{L^2(\Omega^*)} \\ &\leq \underbrace{\widetilde{C}\sqrt{\ell}\,\delta}_{\to \infty} + \underbrace{\|u^{(\ell)} - u\|_{L^2(\Omega^*)}}_{\to 0 \ (\because \text{ Theorem } 5)}, \end{aligned}$$

$\ell \to \infty$:

we have to choose ℓ well.

Theorem 8 (I.)

Let the assumption (♯) in Theorem 5 holds.
We suppose that φ^δ and ψ^δ are real-valued functions with

$$\|\varphi^\delta - \varphi\|_{L^2(\Gamma_0)} \leq \delta, \quad \|\psi^\delta - \psi\|_{L^2(\Gamma_0)} \leq \delta. \tag{4.3}$$

Let $u^{(\ell),\delta}$ be an approximation obtained by the iterative procedure with $\eta^{(0)} = 0$ and $\gamma = \rho$ for Cauchy data φ^δ and ψ^δ. Then, for $0 < \delta \leq 1/e^3$ and $\rho \in [\rho_*, \rho^*]$, we have

$$\|u^{(\ell(\delta,\rho)),\delta} - u\|^2_{L^2(\Omega^*)} \leq \widetilde{C} \begin{cases} \left(\delta^2 \log \frac{1}{\delta}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} & \text{for } \rho_* \leq \rho < \frac{\rho_*}{\rho^*}, \\ \left(\delta^2 \log \frac{1}{\delta}\right)^{\frac{1}{2}} & \text{for } \frac{\rho_*}{\rho^*} \leq \rho \leq \rho^*, \end{cases} \tag{4.4}$$

where $\ell(\delta,\rho)$ is the minimum integer satisfying $\ell(\delta,\rho) \geq \ell_0(\delta,\rho)$ with

$$\ell_0(\delta,\rho) := \begin{cases} \left(\frac{1}{\delta}\right)^{\frac{2\log(1/\rho)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} \left(\log \frac{1}{\delta}\right)^{\frac{\log(\rho^*/\rho_*)}{\log(1/\rho)+\log(\rho^*/\rho_*)}} & \text{for } \rho_* \leq \rho < \frac{\rho_*}{\rho^*}, \\ \frac{1}{\delta}\left(\log \frac{1}{\delta}\right)^{\frac{1}{2}} & \text{for } \frac{\rho_*}{\rho^*} \leq \rho \leq \rho^*, \end{cases}$$

and a positive constant $\widetilde{C}$ depends only on M_0, M, ρ_* and ρ^*.

Outline of the proof

Theorem 5 (I.)

$$
(\sharp)\left(\begin{array}{l}
\textit{Given positive numbers } M_0 \textit{ and } M. \\
\textit{We suppose that } \varphi \textit{ and } \psi \textit{ are real-valued functions} \\
\textit{with } \|\varphi\|^2_{L^2(\Gamma_0)} + \|\psi\|^2_{L^2(\Gamma_0)} \leq M_0^2. \\
\textit{We assume that there exists the solution } u \\
\textit{to the Cauchy problem (1.1) in } \Omega_* \textit{ and its trace on } \Gamma_* \textit{ is in } L^2. \\
\textit{We suppose that } \|u\|_{L^2(\Gamma_*)} \leq M.
\end{array}\right)
$$

Let $u^{(\ell)}$ be an approximate solution obtained by the iterative procedure with $\eta^{(0)} = 0$ and $\gamma = \rho$. Then, for $\ell \geq 2$ and $\rho \in [\rho_*, \rho^*]$, the following estimate holds:

$$
\|u^{(\ell)} - u\|^2_{L^2(\Omega^*)} \leq C \left(\frac{\log \ell}{\ell} \right)^{\min\left\{ \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1 \right\}}, \tag{4.1}
$$

where a positive constant C depends only on M_0, M, ρ_* and ρ^*.

$$\|\varphi\|^2_{L^2(\Gamma_0)} + \|\psi\|^2_{L^2(\Gamma_0)} \leq M_0^2, \quad \|u\|_{L^2(\Gamma_*)} \leq M.$$

$$\|u^{(\ell)} - u\|^2_{L^2(\Omega^*)} \leq C \left(\frac{\log \ell}{\ell} \right)^{\min\left\{ \frac{\log(\rho^*/\rho_*)}{\log(1/\rho)}, 1 \right\}}. \tag{4.1}$$

$$\|u^{(\ell)} - u\|^2_{L^2(\Omega_*)} \leq 8C_1(I_1 + I_2), \tag{5.1}$$

where

$$I_1 := \sum_{k=1}^{\infty} \left\{ 1 - \frac{4}{(\rho^k + \rho^{-k})^2} \right\}^{2\ell} (\rho^*)^{-2k+2} \left| \frac{\varphi_k}{2} - \frac{\psi_k}{2k} \right|^2$$

and

$$I_2 := \sum_{k=1}^{\infty} \left\{ 1 - \frac{4}{(\rho^k + \rho^{-k})^2} \right\}^{2\ell} \left| \frac{\varphi_k}{2} + \frac{\psi_k}{2k} \right|^2 \rho^{2k}.$$

$$\|\varphi\|^2_{L^2(\Gamma_0)} + \|\psi\|^2_{L^2(\Gamma_0)} \le M_0^2, \quad \|u\|_{L^2(\Gamma_*)} \le M.$$

$$I_1 := \sum_{k=1}^{\infty} \left\{1 - \frac{4}{(\rho^k + \rho^{-k})^2}\right\}^{2\ell} (\rho^*)^{-2k+2} \left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2.$$

Lemma 9

If real-valued functions φ and ψ satisfy $\|\varphi\|^2_{L^2(\Gamma_0)} + \|\psi\|^2_{L^2(\Gamma_0)} \le M_0^2$, then we have

$$\sum_{k=1}^{\infty} \left|\frac{\varphi_k}{2} \pm \frac{\psi_k}{2k}\right|^2 \le \frac{M_0^2}{8\pi} =: \frac{1}{2}\widetilde{M_0}^2. \tag{5.2}$$

If real-valued functions φ and ψ satisfy $\|u\|^2_{L^2(\Gamma_*)} \le M^2$, then we have

$$\sum_{k=1}^{\infty} \left|\frac{\varphi_k}{2} - \frac{\psi_k}{2k}\right|^2 \rho_*^{-2k} \le \frac{M^2}{2\pi\rho_*} + \widetilde{M_0}^2 =: \widetilde{M}^2. \tag{5.3}$$

$$\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2}\pm\frac{\psi_k}{2k}\right|^2\le\frac{M_0^2}{8\pi}=:\frac{1}{2}\widetilde{M_0}^2. \tag{5.2}$$

$$\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2\rho_*^{-2k}\le\frac{M^2}{2\pi\rho_*}+\widetilde{M_0}^2=:\widetilde{M}^2. \tag{5.3}$$

$$I_1:=\sum_{k=1}^{\infty}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}(\rho^*)^{-2k+2}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2.$$

Let $0<\lambda<\widetilde{M}$ be given. Let N be the minimum integer satisfying $\left(\frac{\rho_*}{\rho^*}\right)^N\widetilde{M}\le\lambda$, namely $N-1<\frac{\log(\lambda/\widetilde{M})}{\log(\rho_*/\rho^*)}\le N$.

We now divide I_1 into two parts:

$$I_1=\sum_{k=1}^{N-1}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}(\rho^*)^{-2k+2}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2$$
$$+\sum_{k=N}^{\infty}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}(\rho^*)^{-2k+2}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2.$$

$$\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2}\pm\frac{\psi_k}{2k}\right|^2\le\frac{M_0^2}{8\pi}=:\frac{1}{2}\widetilde{M_0}^2. \tag{5.2}$$

$$\sum_{k=1}^{\infty}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2\rho_*^{-2k}\le\frac{M^2}{2\pi\rho_*}+\widetilde{M_0}^2=:\widetilde{M}^2. \tag{5.3}$$

$$I_1:=\sum_{k=1}^{\infty}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}(\rho^*)^{-2k+2}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2.$$

Let $0<\lambda<\widetilde{M}$ be given. Let N be the minimum integer satisfying $\left(\frac{\rho_*}{\rho^*}\right)^N\widetilde{M}\le\lambda$, namely $N-1<\frac{\log(\lambda/\widetilde{M})}{\log(\rho_*/\rho^*)}\le N$.

We now divide I_1 into two parts:

$$I_1=\sum_{k=1}^{N-1}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}(\rho^*)^{-2k+2}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2$$

$$+\sum_{k=N}^{\infty}\left\{1-\frac{4}{(\rho^k+\rho^{-k})^2}\right\}^{2\ell}(\rho^*)^{-2k+2}\left|\frac{\varphi_k}{2}-\frac{\psi_k}{2k}\right|^2.$$

We obtain

$$I_1 \leq C_1 \lambda^{-\mu} \left(\frac{e^a \lambda^{-b} - e^{-a} \lambda^b}{e^a \lambda^{-b} + e^{-a} \lambda^b} \right)^{4\ell} + \lambda^2 =: F(\lambda), \tag{5.4}$$

where

$$C_1 = \frac{1}{2} \widetilde{M_0}^2 \left(\frac{1}{\rho^*} \right)^{\frac{2 \log \widetilde{M}}{\log(\rho^*/\rho_*)}}, \quad \mu = \frac{2 \log(1/\rho^*)}{\log(\rho^*/\rho_*)} > 0,$$

$$b = \frac{\log(1/\rho)}{\log(\rho^*/\rho_*)} > 0, \quad a = b \log \widetilde{M}.$$

Now, let us choose λ such that $F(\lambda)$ is as small as possible.
We choose λ_0 such that

$$\left(\frac{e^a \lambda_0^{-b} - e^{-a} \lambda_0^b}{e^a \lambda_0^{-b} + e^{-a} \lambda_0^b} \right)^{4\ell} = \ell^{-\omega}.$$

that is, we define

$$\lambda_0 = \left\{ \frac{e^{2a} \left(1 - \ell^{-\frac{\omega}{4\ell}} \right)}{1 + \ell^{-\frac{\omega}{4\ell}}} \right\}^{\frac{1}{2b}} = \widetilde{M} \left(\frac{1 - J}{1 + J} \right)^{\frac{1}{2b}}, \tag{5.5}$$

where we put $J := \ell^{-\frac{\omega}{4\ell}}$ for simplicity and we define $\omega > 0$ later.

$$I_1 \le C_1 \lambda^{-\mu} \left(\frac{e^a \lambda^{-b} - e^{-a} \lambda^b}{e^a \lambda^{-b} + e^{-a} \lambda^b} \right)^{4\ell} + \lambda^2 =: F(\lambda). \tag{5.4}$$

$$\lambda_0 = \widetilde{M} \left(\frac{1-J}{1+J} \right)^{\frac{1}{2b}}, \quad \text{where } J := \ell^{-\frac{\omega}{4\ell}}. \tag{5.5}$$

Since we have

$$\frac{1-J}{1+J} = \frac{\omega}{8} \frac{\log \ell}{\ell} \{1 + o(1)\},$$

$$\lambda_0 = \widetilde{M} \left(\frac{\omega}{8} \right)^{\frac{1}{2b}} \left(\frac{\log \ell}{\ell} \right)^{\frac{1}{2b}} \{1 + o(1)\}^{\frac{1}{2b}}.$$

holds. Hence, we get

$$F(\lambda_0) = C_1 \widetilde{M}^{-\mu} \left(\frac{\omega}{8} \right)^{-\frac{\mu}{2b}} \ell^{-\omega + \frac{\mu}{2b}} (\log \ell)^{-\frac{\mu}{2b}} \{1 + o(1)\}$$

$$+ \widetilde{M}^2 \left(\frac{\omega}{8} \right)^{\frac{1}{b}} \left(\frac{\log \ell}{\ell} \right)^{\frac{1}{b}} \{1 + o(1)\}.$$

Since we can choose sufficiently large ω, we obtain

$$I_1 \le F(\lambda_0) \le C_2 \left(\frac{\log \ell}{\ell} \right)^{\frac{1}{b}}$$

for $\ell \ge 2$, where C_2 depends only on M, M_0, ρ_* and ρ^*.