Asymptotics of IDS for the random point interactions

Takuya Mine (Kyoto Institute of Technology)
joint work with
Masahiro Kaminaga (Tohoku Gakuin University)
Fumihiko Nakano (Tohoku University)

March 4th, 2024
Himeji Conference on Partial Differential Equation

Integrated density of states (1)

Consider the random Schrödinger operator

$$H = -\Delta + V_\omega(x) \quad \text{on } \mathbf{R}^d,$$

(ω is some random parameter), and assume that V_ω satisfies some stastistically translational invariant condition (e.g., $V_\omega(x)$ and $V_\omega(x+a)$ have the same probability distribution). Then, for $\lambda \in \mathbf{R}$, the integrated density of states (IDS) $N(\lambda)$ is defined as follows:

$$N(\lambda) = \lim_{L \to \infty} \frac{N_{Q_L}(\lambda)}{|Q_L|},$$
$$N_{Q_L}(\lambda) = \#\{\mu : \text{eigenvalue of } H_{Q_L}; \mu \le \lambda\},$$

where H_{Q_L} is the operator H restricted on the cube $Q_L = [0, L]^d$ with some boundary conditions, and $|S|$ is the Lebesgue measure of the set S.

Integrated density of states (2)

It is widely believed that IDS is closely related to the electric conductance of the material. If the density of states (derivative of IDS) is small in some interval I, then an electron with energy in I can hardly move along the material (insulator).

In the mathematical analysis of IDS, one of the main topic is the asymptotics of IDS near the bottom of the spectrum of H.

Lifshitz tail (1)

Consider the Anderson model

$$V_\omega = \sum_{n \in \mathbf{Z}^d} a_\omega(n) V_0(x-n), \quad V_0 \in C_0^\infty(\mathbf{R}^d; \mathbf{R}) \setminus \{0\},$$

$$\operatorname{supp} V_0 \subset [0,1]^d, \ \{a_\omega(n)\}_{n \in \mathbf{Z}^d} : \text{i.i.d.}, \ a_\omega(n) \geq 0.$$

If $V_0 \geq 0$, under appropriate condition on the probability distribution of $a_\omega(n)$, it is known that $\sigma(H) = [0, \infty)$ and the Lifshitz tail

$$N(\lambda) \leq e^{-C\lambda^{-d/2}} \quad (\lambda > 0)$$

holds (cf. Lifshitz 1965, Nakao 1977, ...). There is a nice Japanese review by Ueki 2014 about this subject.

Lifshitz tail (2)

If $V_0 \leq 0$, then the spectrum can have the negative part: $\sigma(H) \supset [\lambda_0, \lambda_0 + \epsilon]$, $\lambda_0 = \inf \sigma(H) < 0$. Even in this case, the Lifshitz tail

$$N(\lambda) \leq e^{-C(\lambda - \lambda_0)^{-d/2}} \quad (\lambda_0 < \lambda < \lambda_0 + \epsilon)$$

holds under appropritate conditions (cf. Klopp 2002).

Random positions

Next, consider the case that the positions of obstacles are random:

$$V_\omega(x) = \sum_{y \in Y_\omega} V_0(x - y),$$

where Y_ω is some random set obeying some probability distribution. The random point measure

$$\mu_\omega = \sum_{y \in Y_\omega} \delta_y$$

is called a point process. Y_ω is regarded as the support of μ_ω.

Recently, the theory of point process is extensively studied from the viewpoint of the probability theory, the statistics, and various practical applications.

Examples of point processes (1)

(1) Poisson point process. Most basic point process, which represents the complete spatial randomness. (Today's main topic)

(2) Gibbs point process. The point process having the Radon-Nykokim density with respect to the Poisson point process with intensity 1. Many point processes are represented in this form, e.g., Hard core process, Strauss process, etc. (cf. Nakagawa 2023)

(3) Cox point process. Poisson point process with random intensity measure.

(4) Determinantal point process or Fermion point process. Random points have some repulsive interactions (random points tend to escape from each other). (cf. Japanese review by Shirai 2014)

Examples of point processes (2)

The book

'Spatial Point Patterns, Methodology and Applications with R'
by Adrian Baddeley, Ege Rubak, and Rolf Turner

contains much more examples, and explain how to simulate point processes by using the R library `spatstat`.

Below we shall show some pictures of point processes created by `spatstat`.

Poisson point process (1)

We say μ_ω is the Poisson point process with intensity measure ρdx ($\rho > 0$ is a constant) if the following holds.

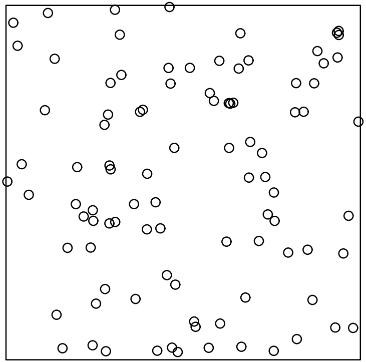
(1) For any bounded measureble set S, $\mu_\omega(S)$ obeys the Poisson distribution with parameter $\rho|S|$, where $|S|$ is the Lebesgure measure of S.

(2) For any disjoint bounded measurable sets S_j $(j = 1, \ldots, n)$, the random variables $\mu_\omega(S_j)$ $(j = 1, \ldots, n)$ are independent.

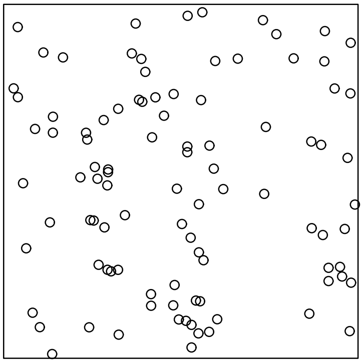
In the next slide, we show two examples of the Poisson point process on $[0, 10]^2$ with intensity $1dx$.

Poisson point process (2)

Poisson point process



Poisson point process



Hard core point process (1)

The hard core point process is obtained by removing the events

'there is at least a pair $x, y \in Y_\omega$ with $|x - y| \leq R$'

from the Poisson point process. Consequently, the balls

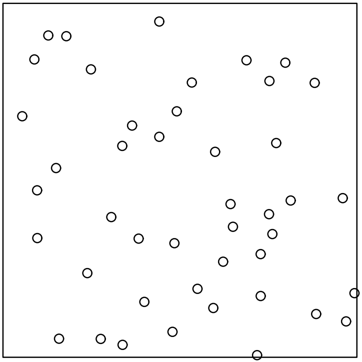
$$\{B_y(R/2)\}_{y \in Y_\omega}, \quad B_y(R) = \{x \in \mathbf{R}^d; |x - y| < R\}$$

do not intersect with each other.

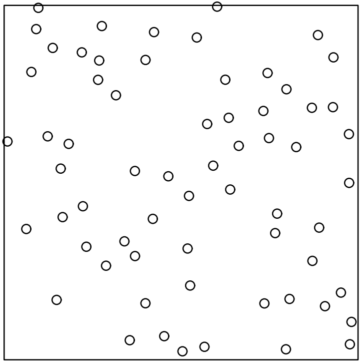
In the next slide, we show two examples of the hard core point process on $[0, 10]^2$ with (original) intensity $1dx$ and $R = 0.5$.

Hard core point process ($R = 0.5$)

Hard core point process R 0.5



Hard core point process R 0.5



Determinantal point process (1)

The determinantal point process with Gaussian kernel is characterized by the kernel function

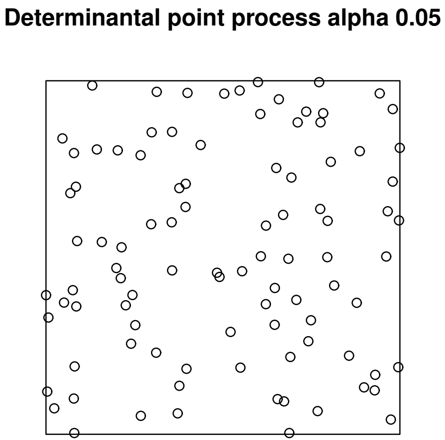
$$C(x,y) = \lambda R(x,y), \quad R(x,y) = \exp\left(-\frac{|x-y|^2}{\alpha^2}\right),$$

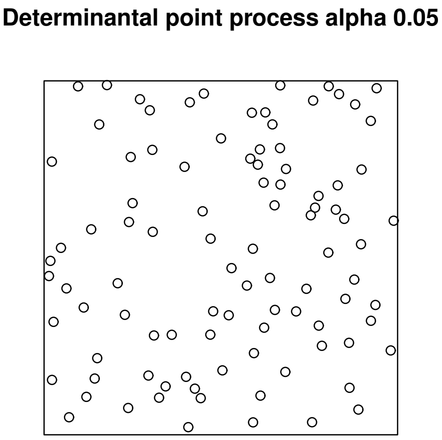
where $\lambda > 0$ and $\alpha > 0$ are constants with $\lambda \le 1/(\pi\alpha^2)$. The intensity is λdx, and the n-point correlation function is given by

$$\rho(x_1, \ldots, x_n) = \det(R(x_i, x_j))_{i,j=1}^n.$$

If α is large, then the points tend to escape from other points (repulsive interaction).
In the next slides, we show two examples of the determinantal point process with Gaussian kernel on $[0,1]^2$ with $\lambda = 100$, $\alpha = 0.05$ or $\alpha = 0.001$.

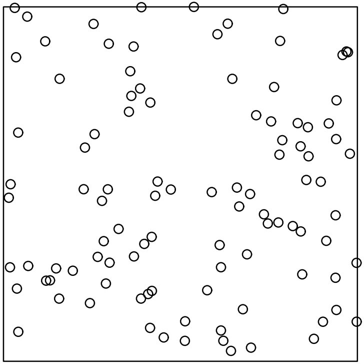
Determinantal point process ($\alpha = 0.05$)



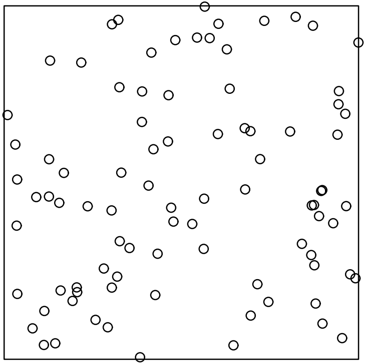


Determinantal point process ($\alpha = 0.001$)

Determinantal point process alpha 0.001



Determinantal point process alpha 0.001



Poisson random potential (1)

Consider the random potential

$$V_\omega(x) = \sum_{y \in Y_\omega} V_0(x - y),$$

$V_0 \in C_0^\infty(\mathbf{R}^d)$, $V_0 \le 0$, and the minimum of V_0 is $V_0(0) < 0$. Assume Y_ω is the support of the Poisson point process with constant intensity measure ρdx ($\rho > 0$).

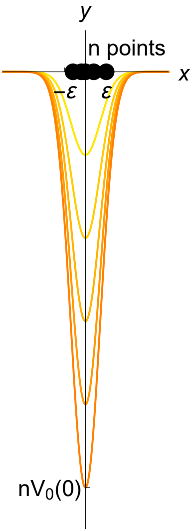
Poisson random potential (2)

In this case, the spectrum becomes the whole real line

$$\sigma(H) = \mathbf{R},$$

even if V_0 is bounded from below.

If n random points exist near 0, the depth of the potential well is almost multiplied by n. The number n can be arbitrarily large, so the potential well can be arbitrarily deep (with very small probability).



Pastur tail (1)

The above mechanism also explains very rapid decay of IDS as $\lambda \to -\infty$:

$$\log N(\lambda) = -\frac{|\lambda|}{|V_0(0)|} \log|\lambda| \cdot (1 + o(1)) \quad \text{as } \lambda \to -\infty \tag{1}$$

(Pastur 1974, 1977).

(1) means $N(\lambda)$ decays super exponentially $O(|\lambda|^{-C|\lambda|})$ as $\lambda \to -\infty$, and is sometimes called the Pastur tail.

Pastur tail (2)

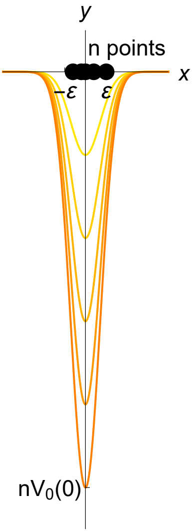
When $V_0 \leq 0$, and $V_0(0)$ is the minimum, negative spectrum λ is created by at least

$$n = \frac{|\lambda|}{|V_0(0)|}$$

random points in a small ball B_ϵ. The probability of this event is

$$p \doteqdot e^{-\rho|B_\epsilon|} \frac{(\rho|B_\epsilon|)^n}{n!}.$$

Combining this estimate with the Stirling formula $n! \sim (2\pi n)^{1/2} (n/e)^n$, we get the Pastur tail (1).



Schrödinger operator with point interactions

Next we introduce the point interactions.

Let Y be a locally finite set in $\mathbf{R}^d$ $(d = 1, 2, 3)$, that is,

$$\#(Y \cap B_0(R)) < \infty$$

for every $R > 0$, where $B_x(R) = \{y \in \mathbf{R}^d \mid |y - x| < R\}$. Let $\alpha = (\alpha_y)_{y \in Y}$ be a sequence of real numbers. We consider the Schrödinger operator $-\Delta_{\alpha,Y}$, formally written as

$$-\Delta_{\alpha,Y} = -\Delta + \text{`point interactions on } Y\text{'},$$

where α_y is the parameter representing the interaction at the point y. Basic facts about $-\Delta_{\alpha,Y}$ are found in the book 'Solvable models in quantum mechanics' by Albeverio et al.

Definition of Point interactions

A rigorous definition of $-\Delta_{\alpha,Y}$ is as follows.

$$-\Delta_{\alpha,Y}u = -\Delta|_{\mathbf{R}^d\setminus Y}u,$$

$$D(-\Delta_{\alpha,Y}) = \{u \in H^2_{\mathrm{loc}}(\mathbf{R}^d \setminus Y) \cap L^2(\mathbf{R}^d)\,;\, -\Delta|_{\mathbf{R}^d\setminus Y}u \in L^2(\mathbf{R}^d),$$

$$u \text{ satisfies } (BC)_y \text{ for every } y \in Y\}.$$

Here, $-\Delta|_{\mathbf{R}^d\setminus Y}u$ is defined as a Schwartz distribution on $\mathbf{R}^d \setminus Y$.
The boundary condition $(BC)_y$ is as follows:

$\boxed{d=1}$ $u(y+) = u(y-) = u(y)$, $u'(y+) - u'(y-) = \alpha_y u(y)$.

$\boxed{d=2}$ $u(x) = u_{y,0}\log|x-y| + u_{y,1} + o(1)$ as $x \to y$, and $2\pi\alpha_y u_{y,0} + u_{y,1} = 0$.

$\boxed{d=3}$ $u(x) = u_{y,0}|x-y|^{-1} + u_{y,1} + o(1)$ as $x \to y$, and $-4\pi\alpha_y u_{y,0} + u_{y,1} = 0$.

Spectrum of $-\Delta_{\alpha,Y}$ for finite Y

The following result is taken from the book of Albeverio et al.

Proposition 1 (Spectrum of $-\Delta_{\alpha,Y}$ for finite Y)

Let $d = 3$. Let $Y = \{y_j\}_{j=1}^N$ be a finite set and $\alpha = (\alpha_j)_{j=1}^N$ (we write $\alpha_j = \alpha_{y_j}$). Then, for $\lambda = -s^2$ $(s > 0)$, λ is an eigenvalue of $-\Delta_{\alpha,Y}$ if and only if $\det A = 0$, where $A = (a_{jk})$ is the $N \times N$ matrix given by

$$a_{jk} = \begin{cases} \alpha_j + \dfrac{s}{4\pi} & (j = k), \\ -\dfrac{e^{-s|y_j - y_k|}}{4\pi|y_j - y_k|} & (j \neq k). \end{cases}$$

The non-diagonal component a_{jk} is $-G(y_j, y_k; -s^2)$, where $G(x, x'; \lambda)$ is the integral kernel of $(-\Delta - \lambda)^{-1}$. There are similar formulas in the case $d = 1, 2$.

Spectrum of $-\Delta_{\alpha,Y}$ for $\#Y = 1$

In the case $\#Y = 1$ and $\alpha_1 = \alpha$, $\lambda = -s^2$ ($s > 0$) is an eigenvalue of $-\Delta_{\alpha,Y}$ if and only if

$$\alpha + \frac{s}{4\pi} = 0.$$

Thus

$$\sigma(-\Delta_{\alpha,Y}) \cap (-\infty, 0) = \begin{cases} \{-(4\pi\alpha)^2\} & (\alpha < 0), \\ \emptyset & (\alpha \geq 0). \end{cases}$$

Spectrum of $-\Delta_{\alpha,Y}$ for $\#Y = 2$ (1)

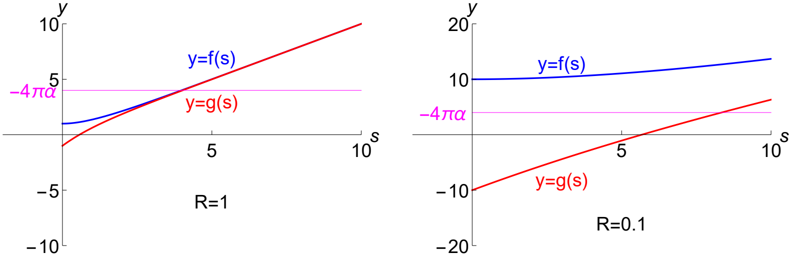
In the case $\#Y = 2$, $|y_1 - y_2| = R$ and $\alpha_1 = \alpha_2 = \alpha$, $\lambda = -s^2$ ($s > 0$) is an eigenvalue of $-\Delta_{\alpha,Y}$ if and only if 0 is an eigenvalue of

$$A = \begin{pmatrix} \alpha + \dfrac{s}{4\pi} & -\dfrac{e^{-sR}}{4\pi R} \\ -\dfrac{e^{-sR}}{4\pi R} & \alpha + \dfrac{s}{4\pi} \end{pmatrix},$$

that is,

$$\alpha + \frac{1}{4\pi}\left(s + \frac{e^{-sR}}{R}\right) = 0 \Leftrightarrow f(s) := s + \frac{e^{-sR}}{R} = -4\pi\alpha,$$
$$\alpha + \frac{1}{4\pi}\left(s - \frac{e^{-sR}}{R}\right) = 0 \Leftrightarrow g(s) := s - \frac{e^{-sR}}{R} = -4\pi\alpha.$$

Spectrum of $-\Delta_{\alpha,Y}$ for $\#Y = 2$ (2)



As $R \to +0$, the y-intercept $-1/R$ of $y = g(s)$ tends to $-\infty$. Thus the solution s of $g(s) = -4\pi\alpha$ tends to ∞ as $R \to +0$, so $\lambda = -s^2 \to -\infty$.

Thus we see the following:

'When $\#Y = 2$ and two points becomes closer, then an eigenvalue of $-\Delta_{\alpha,Y}$ tends to $-\infty$'.

Thus the behavior of the eigenvalues are completely different from the scalar potential case.

Spectrum of $-\Delta_{\alpha,Y}$ for $\#Y = 2$ (3)

For fixed s, let $R = R_\alpha(s)$ be the solution of

$$s - \frac{e^{-sR}}{R} = -4\pi\alpha$$

with respect to R. If $\alpha = 0$, then the equation is simplified as

$$sR - e^{-sR} = 0 \Leftrightarrow sR = t_0,$$

where t_0 is the unique solution of $t - e^{-t} = 0$ $\left(t_0 \doteqdot 0.5671\ldots\right)$. Thus we have explicitly

$$R_0(s) = \frac{t_0}{s}.$$

Even if $\alpha \neq 0$, we have a similar asymptotics

$$R_\alpha(s) \sim \frac{t_0}{s} \quad (s \to \infty),$$

where $f \sim g$ means $f/g \to 1$.

Poisson point interaction (1)

Let us consider the Schrödinger operators with Poisson point interactions, that is, the operator $-\Delta_{\alpha,Y_\omega}$ when Y_ω is the Poisson configuration (support of the Poisson point process).

For simplicity, we assume α is a constant sequence (the value α_y is independent of y), and we denote its common value also by α. The basic facts about this operator are summarized as follows.

Poisson point interaction (2)

Theorem 2 (Kaminaga–M–Nakano 2020)

Let $d = 1, 2, 3$. Assume α is a constant sequence, and Y_ω is the Poisson configuration with intensity ρdx, where $\rho > 0$ is a constant. Then, the following holds.

(1) The operator $-\Delta_{\alpha, Y_\omega}$ is self-adjoint almost surely.

(2) We have almost surely

$$\sigma(-\Delta_{\alpha, Y_\omega}) = \begin{cases} [0, \infty) & (\alpha \geq 0), \\ \mathbf{R} & (\alpha < 0), \end{cases} \quad (d = 1),$$

$$\sigma(-\Delta_{\alpha, Y_\omega}) = \mathbf{R} \quad (\alpha \in \mathbf{R}), \quad (d = 2, 3).$$

The result for $d = 1$ is obtained in Minami 1988. The result (2) for $d = 2, 3$ comes from the fact the point interaction is always negative when $d = 2, 3$.

Asymptotics of IDS as $\lambda \to -\infty$ ($d = 1$, $\alpha < 0$)

When $d = 1$ and $\alpha < 0$, then $\sigma(-\Delta_{\alpha,Y_\omega}) = \mathbf{R}$, and we can consider the asymptotics of IDS as $\lambda \to -\infty$. The result is written in the book 'Introduction to the theory of disordered systems' by Lifshits–Gredskul–Pastur, as follows.

$$N(\lambda) \sim \sqrt{|\lambda|} \exp\left[-2|\alpha|^{-1}|\lambda|^{1/2} \log\left(\frac{|\lambda|}{|\alpha|\rho}\right)\right] \text{ as } \lambda \to -\infty$$

(Magarill–Entin 1966).

The decay order is a bit weaker than exponential decay, but faster than polynomial decay.

Main result for $d = 3$ (1)

Theorem 3 (Kaminaga–Nakano–M, preprint.)

Let $d = 3$. Suppose that α is a constant sequence, and Y_ω is the Poisson configuration with intensity measure ρdx, $\rho > 0$. Let $N(\lambda)$ be the corresponding IDS for $-\Delta_{\alpha,Y_\omega}$. Then we have

$$N(-s^2) = \frac{2\pi}{3}\rho^2 R_\alpha(s)^3 + O(s^{-6+\epsilon}) \quad (s \to \infty), \tag{2}$$

for every $0 < \epsilon < 3$, where $R_\alpha(s)$ is the solution of

$$s - \frac{e^{-sR}}{R} = -4\pi\alpha$$

with respect to R.

Main result for $d = 3$ (2)

Theorem 3 (continued)

In particular, the principal term is

$$N(-s^2) \sim \frac{2\pi}{3}\rho^2 t_0^3 s^{-3} \quad (s \to \infty), \tag{3}$$

where t_0 is the unique positive solution of $t = e^{-t}$ ($t_0 \doteqdot 0.567$).

Theorem 3 says

$$N(\lambda) = O(|\lambda|^{-3/2}) \quad (\lambda \to -\infty).$$

The principal term given in (3) is independent α, but the first term in RHS of (2) gives more accurate approximation, according to our numerical verification.

Numerical test

In order to verify the result

$$N(-s^2) \sim \frac{2\pi}{3}\rho^2 R_\alpha(s)^3 \quad (s \to \infty),$$

we'll calculate the IDS numerically in the following way.

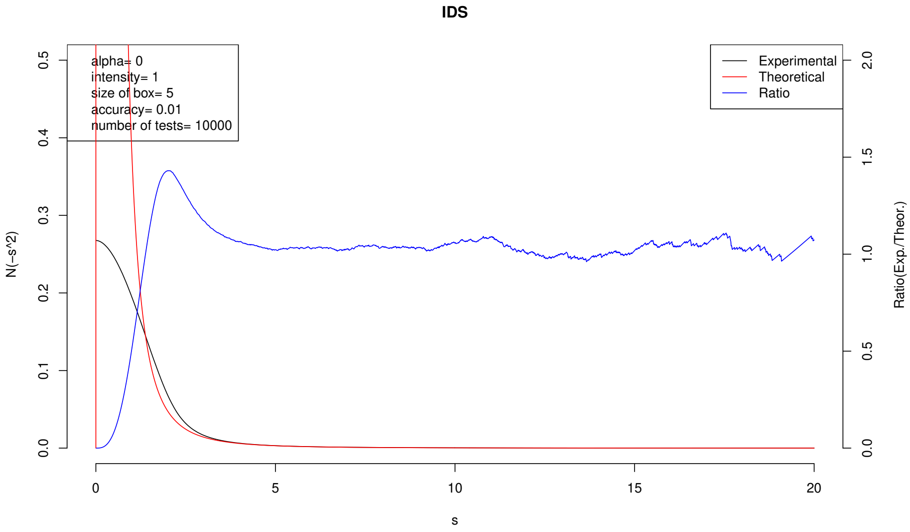
(i) Generate a sample Poisson configuration with intensity 1 in the box $Q = [0, L]^3$.

(ii) Calculate the eigenvalue counting function $N_Q(-s^2)$, by calculating $\det A(s)$ where $A(s)$ is given in Proposition 1.

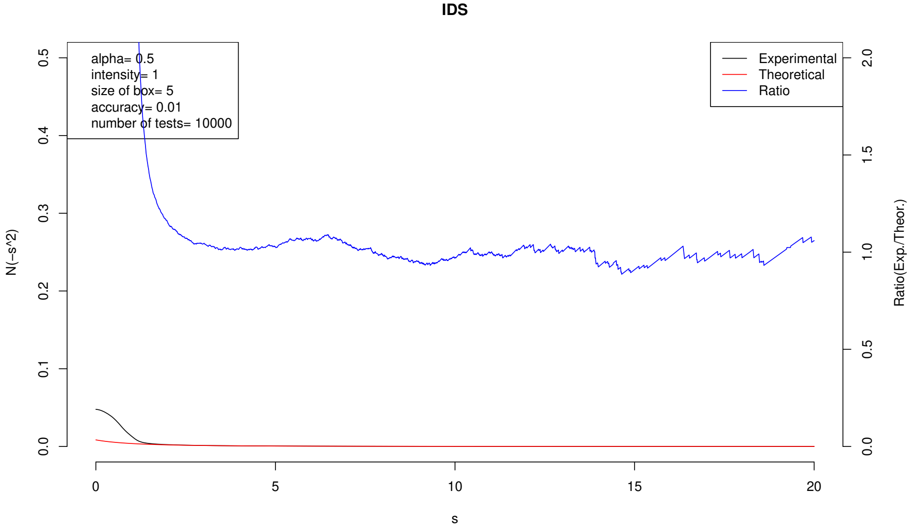
(iii) Repeat (i) and (ii) many times, calculate the average, and divide it by L^3. Then we get $\mathbf{E}[N_Q(-s^2)]/L^3 \doteqdot N(-s^2)$.

This time we take $L = 5$, and repeat the tests 10000 times. We show the result in the following slides.

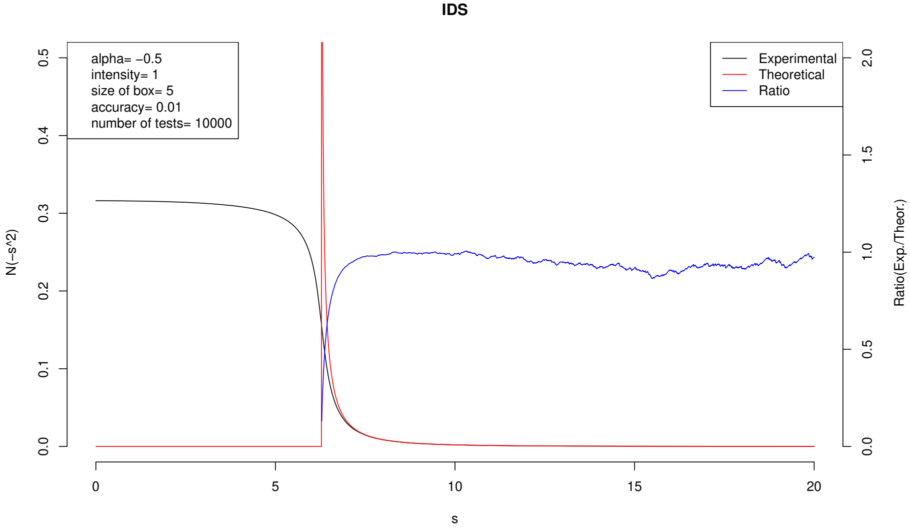
Numerical result ($\alpha = 0.0$, 10000 tests)



Numerical result ($\alpha = 0.5$, 10000 tests)



Numerical result ($\alpha = -0.5$, 10000 tests)



Conjecture for $d = 2$ (1)

Conjecture 4

Let $d = 2$. Suppose that α is a constant sequence, and Y_ω is the Poisson configuration with intensity measure ρdx, $\rho > 0$. Let $N(\lambda)$ be the corresponding IDS for $-\Delta_{\alpha,Y_\omega}$. Then we have

$$N(-s^2) = \frac{\pi}{2}\rho R_\alpha(s)^2 + O(s^{-8+\epsilon}) \quad (s \to \infty), \tag{4}$$

for every $0 < \epsilon < 4$, where $R_\alpha(s)$ is the solution of

$$2\pi\alpha + \gamma_E + \log\frac{s}{2} - K_0(sR) = 0, \tag{5}$$

where γ_E is the Euler constant, and K_0 is 0-th order modified Bessel function of the 2nd kind.

Conjecture for $d = 2$ (2)

The modified Bessel function comes from the integral kernel $(-\Delta + s^2)^{-1}(x, y)$ in $\mathbf{R}^2$. The equation (5) again comes from the eigenvalue equation for $-\Delta_{\alpha,Y}$ in $\mathbf{R}^2$ for $\#Y = 2$.

An asymptotic analysis of (5) tells us

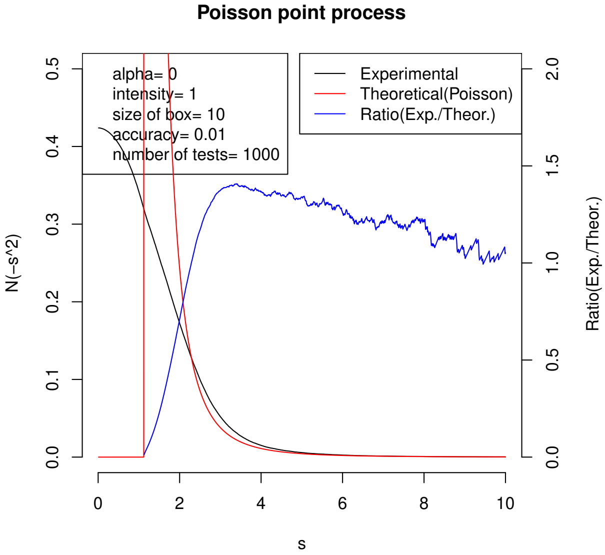
$$R_\alpha(s) = O(s^{-2}) \quad (s \to \infty).$$

So (4) means

$$N(-s^2) = O(s^{-4}) \quad (s \to \infty) \Leftrightarrow N(\lambda) = O(|\lambda|^{-2}) \quad (\lambda \to -\infty).$$

Conjecture 4 seems to be proved similarly (the detail is not completely checked yet). The numerical analysis below also supports Conjecture 4.

Numerical result (Poisson point process)



The blue curve (ratio) seems to converge to 1 as $s \to \infty$, which supports Conjecture 4.

Other point processes

When $d = 2$, we can simulate various point processes, and calculate IDS for that model, by using `spatstat` library.

Below we simulate the following two models.

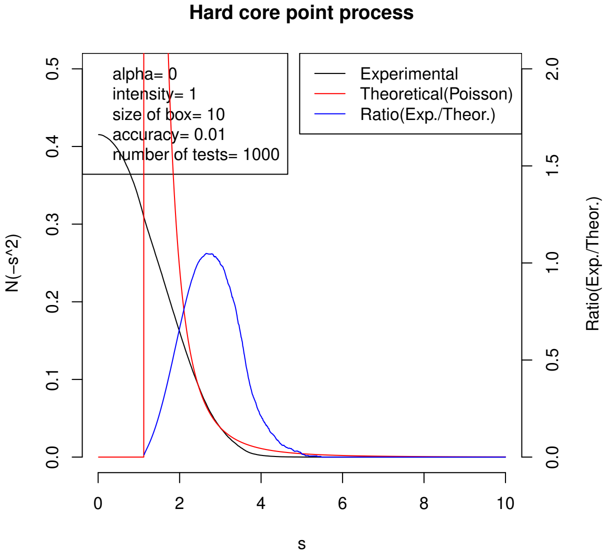
(1) Hard core point process on $[0, 10]^2$, with hard core distance $R = 0.1$.

(2) Determinantal point process on $[0, 1]^2$ with Gaussian kernel, intensity 100 and n-point correlation function

$$\rho(x_1, \ldots, x_n) = \det(R(x_i, x_j))_{i,j=1}^n,$$
$$R(x, y) = \exp\left(-\frac{|x-y|^2}{\alpha^2}\right),$$

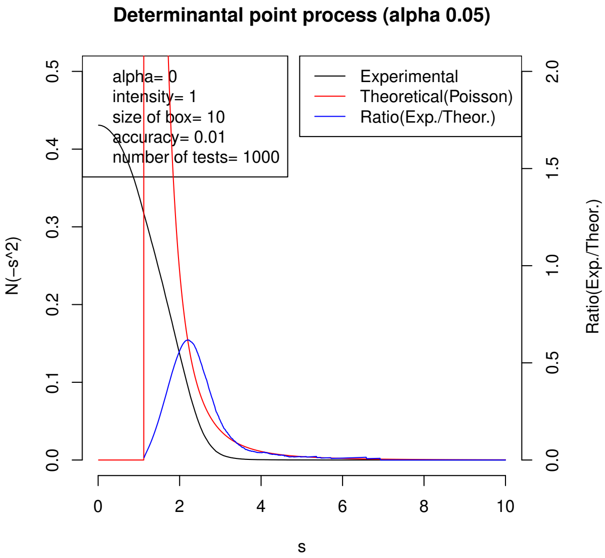
and multiply 10 to the points (scaling). We choose $\alpha = 0.05$ (strong repulsion) and $\alpha = 0.001$ (weak repulsion).

Numerical result (Hard core point process)



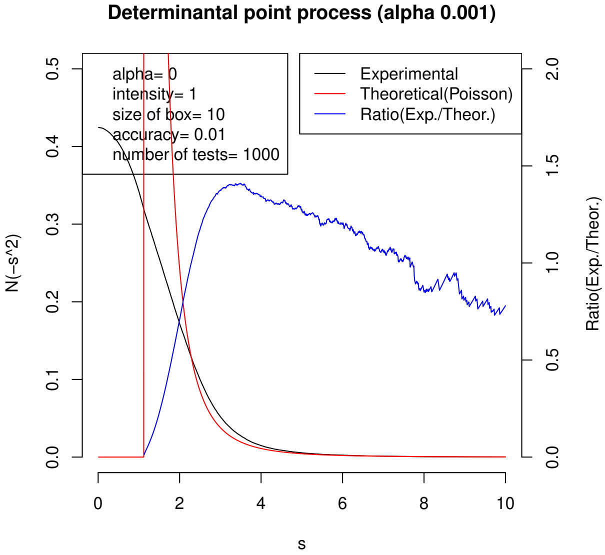
The operator is bounded from below and $N(-s^2) = 0$ for large s, since the distance of two different points ≥ 0.1.

Numerical result (Determinantal Point process)



Strong repulsion $\alpha = 0.05$. The decay of IDS seems faster than the Poisson case.

Numerical result (Determinantal Point process)



Weak repulsion $\alpha = 0.001$. For small α, the decay of IDS seems closer to the Poisson case, at least for $s \leq 6$.

Strategy of Proof of Theorem 3

Recall the calculation of $\sigma(-\Delta_{\alpha,Y})$ when $d = 3$ and $\#Y = 2$. For large $s > 0$, we find an eigenvalue $\doteqdot -s^2$ of $-\Delta_{\alpha,Y}$ if we find a very close pair $\{y_1, y_2\} \subset Y$ with $|y_1 - y_2| = R_\alpha(s) \sim t_0/s$.

Thus the proof consists of the following two steps.

(i) For a cube $Q_L = [0, L]^3$, calculate the expectation of the number of pairs $\{y_1, y_2\} \subset Y \cap Q_L$ with $|y_1 - y_2| \leq R_\alpha(s)$.

(ii) Estimate the difference between the eigenvalue counting function $N_{Q_L}(-s^2)$ and the number of pairs given in (i).

(i) is a problem in the probability theory, and has a definite answer in the case of the Poisson point process.

(ii) can be solved by using Proposition 1, and an analysis of the eigenvalue of the matrix $A(s)$.

Number of close pairs (1)

Let Y_ω be the Poisson configuration in $\mathbf{R}^d$ with intensity measure ρdx. For $y \in Y_\omega$ and $R > 0$, let

$$n_y(R) = \#(Y_\omega \cap B_y(R)), \quad B_y(R) = \{x; |x-y| < R\}.$$

$n_y(R)$ is the number of Poisson points in the ball $B_y(R)$. The point y itself is counted, so $n_y(R) \geq 1$.

Roughly speaking, the number of pairs we need is expressed as

$$\frac{1}{2}\#\{y \in Y_\omega \cap Q_L \,;\, n_y(R_\alpha(s)) = 2\}.$$

Fortunately, the expectation of this number can be calculated explicitly.

Number of close pairs (2)

Proposition 5

Let $d = 1, 2, 3, \ldots$, and Y_ω be the Poisson configuration in $\mathbf{R}^d$ with intensity measure ρdx, where $\rho > 0$ is a constant. For $L > 0$, let $Q_L = [0, L]^d$. Then, we have for $n = 1, 2, 3, \ldots$

$$\frac{\mathbf{E}[\#\{y \in Y_\omega \cap Q_L\,;\, n_y(R) = n\}]}{|Q_L|} = \frac{1}{(n-1)!}|B_0(R)|^{n-1}\rho^n e^{-\rho|B_0(R)|}.$$

In particular, when $d = 3$, $n = 2$, and $R = R_\alpha(s)$, we have

$$\frac{1}{2}\frac{\mathbf{E}[\#\{y \in Y_\omega \cap Q_L\,;\, n_y(R_\alpha(s)) = 2\}]}{|Q_L|} \doteqdot \frac{2\pi R_\alpha(s)^3}{3}\rho^2,$$

which is the first term in (4).

Estimate of the remainder term

Lemma 6

Let $d = 3$. Let α is a constant sequence, Y_ω is the Poisson configuration with intensity ρdx, where $\rho > 0$ is a constant. Then, for every δ with $1/2 < \delta < 1$, and for every $m > 0$, there exist constants $R_0 > 0$ and $C > 0$ such that

$$\begin{aligned}
&\mathbf{E}[N_{Q_L}(-(s_\alpha(R) - R^m))^2] \\
&\geq \frac{1}{2}\mathbf{E}[\#\{y \in Y_\omega \cap Q_L\,;\, n_y(R) = 2\}] - CR^{6\delta}|Q_L|, \\
&\mathbf{E}[N_{Q_L}(-(s_\alpha(R) + R^m))^2] \\
&\leq \frac{1}{2}\mathbf{E}[\#\{y \in Y_\omega \cap Q_L\,;\, n_y(R) = 2\}] + CR^{6\delta}|Q_L|,
\end{aligned}$$

for every $0 < R < R_0$ and every $L > R^{-2}$.

Further problems

1. Can we find the asymptotics of IDS for other point processes? Especially, is there a result corresponding to Proposition 5? The case of the determinantal process with Gaussian kernel seems interesting.
2. For $d = 3$ and $\alpha < 0$, the height of the jump of IDS $N(-s^2)$ around $s = -4\pi\alpha$ (eigenvalue for one-point interaction)? Asymptotics as $\alpha \to -\infty$?
3. Anderson localization (dense pure point spectrum in $(-\infty, 0)$)?
4. Is there absolutely continuous spectrum in $[E_0, \infty)$ for large $E_0 > 0$? (Delocalization conjecture, but very difficult...)

References

1. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics. Second edition. With an appendix by Pavel Exner*, AMS Chelsea Publishing, Providence, RI, 2005.
2. Adrian Baddeley, Ege Rubak, and Rolf Turner, *Spatial Point Patterns, Methodology and Applications with R*, Chapman and Hall/CRC, 2015.
3. M. Kaminaga, T. Mine and F. Nakano, A Self-adjointness Criterion for the Schrödinger Operator with Infinitely Many Point Interactions and Its Application to Random Operators, *Ann. Henri Poincaré* **21**, 405–435, 2020.
4. F. Klopp, Lifshitz tails for random perturbations of periodic Schrödinger operators, *Proc. Indian Acad. Sch. (Math. Sci.)* **112** (1), 147–162.

References

5. I. M. Lifshits, A. S. Gredeskul, and L. A. Pastur, *Introduction to the theory of disordered systems*, John Wiley & Sons, Inc., New York, 1988.
6. 白井朋之, Fermion and boson point processes and related topics, 数理解析研究所講究録 第 1921 巻 (2014), 12–27. (日本語)
7. Y. Nakagawa, Asymptotic behaviors of the integrated density of states for random Schrödinger operators associated with Gibbs point processes, *Electron. J. Probab.* **28** (2023), 1–14.
8. 上木直昌, 確率解析とランダムシュレディンガー作用素, 数学 66 巻 4 号 (2014), 392–412. (日本語) `https://doi.org/10.11429/sugaku.0664392`