Stability of charge density waves in electron-phonon systems

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Based on:
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Part 1: Background

Motivation

- Various phases are anticipated to emerge in many-electron systems.
  Eg. ferromagnetic phase, antiferromagnetic phase, charge density wave phase, Mott insulator, Luttinger liquid, Fermi liquid, etc..
- The mathematical justification of phase diagrams has only been partially achieved.
- Systems with electron-phonon interactions are typically more challenging to handle than ordinary many-electron systems, and there is little mathematical proof of phase diagrams for low-temperature phases.

History

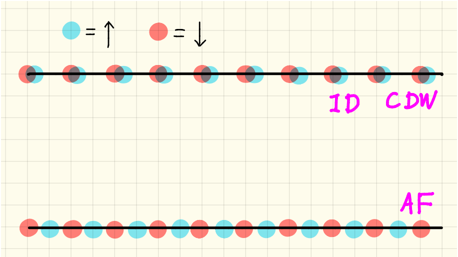
- The original meaning of CDW refers to the periodic structure of charge with a period of $2a$ (where a is the lattice spacing) in one-dimensional electron systems.
  In 1954, H. Fröhlich[1] proposed this mechanism to theoretically explain superconductivity (which was incorrect as a theory of superconductivity).
- In the 1970s, CDWs were discovered in several one-dimensional conductors, and their behavior was experimentally and theoretically elucidated.[2]
- In present times, the term "charge density wave" is commonly used to describe the type of order depicted in the figure, even in multi-dimensional electron systems.

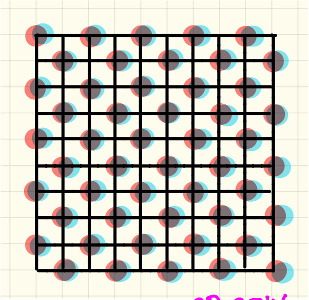
[1] H. Fröhlich. On the Theory of Superconductivity: The One-Dimensional Case. Proceedings of the Royal Society A 223 (1154): 296–305.

[2] G. Grüner, Density Waves in Solids. Addison-Wesley, 1994.

Mathematical investigations of charge density waves

- $D = 1$: Lieb–Nachtergaele [3], T.M. [4], Gontier–Kouande–Sere[5]
- $D \geq 2$: Macrius–Piguet[6], Borgs–Jedrzejewski–Kotecky [7], Borgs–Kotecky[8], T.M. [9]





[3] E. H. Lieb and B. Nachtergaele, Phys. Rev. B 51, 4777, 1995
[4] T. M., Rev. Math. Phys. Vol. 23, No. 07, pp. 749-822 (2011)
[5] D. Gontier, A. E. K. Kouande and E. Sere, A. H. Poincare, Vol. 24, pages 3945–3966, (2023)
[6] N. Macris and C.-A. Piguet, Phys. Rev. B 60, 13484 (1999)
[7] C Borgs, J Jedrzejewski, R Kotecky, J. Phys. A: Math. Gen. 29 733 (1999)
[8] C. Borgs, R. Kotecky, Comm. Math. Phys., Vol. 208, pages 575–604, (2000)
[9] T.M., J. Stat. Phys., Vol. 165, pages 225–245, (2016)

Part 2: The Existence of Charge Density Waves as Long-Range Order

Mathematical description of electrons

- Here I give a mathematical framework for describing electrons on finite lattice $\Lambda = \mathbb{Z}^d \cap [-L, L)^d$. A state of a single electron is represented by a unit vector in the Hilbert space
$$\ell^2(\Lambda \times \{\uparrow, \downarrow\}).$$
- Since electrons are fermions, states of N electrons are expressed as vectors in the antisymmetric tensor product Hilbert space:
$$\mathfrak{F}_{\mathrm{e},N}(\Lambda) = \bigwedge^N \ell^2(\Lambda \times \{\uparrow, \downarrow\}).$$
- It is more convenient to employ the fermionic Fock space to describe states with various numbers of electrons:
$$\mathfrak{F}_{\mathrm{e}}(\Lambda) = \bigoplus_{N=0}^{2|\Lambda|} \mathfrak{F}_{\mathrm{e},N}(\Lambda).$$

- The creation and annihilation operators $\{c_{x,\sigma}, c_{x,\sigma}^* : x \in \Lambda, \sigma = \{\uparrow, \downarrow\}\}$ acting on $\mathfrak{F}_{\mathrm{e}}(\Lambda)$ will be needed to define various operators: [10]

$$\{c_{x\sigma}, c_{y\tau}\} = 0 = \{c_{x\sigma}^*, c_{y\tau}^*\},$$
$$\{c_{x\sigma}, c_{y\tau}^*\} = \delta_{xy}\delta_{\sigma\tau}.$$

- Number operator of electrons:

$$\hat{n}_{x\sigma} = c_{x\sigma}^* c_{x\sigma}, \quad \hat{n}_x = \hat{n}_{x\uparrow} + \hat{n}_{x\downarrow}.$$

[10] A. Arai. Analysis on Fock Spaces and Mathematical Theory of Quantum Fields. WORLD SCIENTIFIC, Dec. 2016.
O. Bratteli and D. W. Robinson. Operator Algebras and Quantum Statistical Mechanics 2: Equilibrium States. Models in Quantum Statistical Mechanics. Springer Berlin Heidelberg, 1997.

Phase diagram in classical systems

Toy model:

$$H_{\mathrm{C},\Lambda} = U \sum_{x\in\Lambda} \hat{n}_{x,\uparrow}\hat{n}_{x,\downarrow} + W \sum_{\langle x,y\rangle\in E_\Lambda} \hat{n}_x \hat{n}_y - \left(\mu + 2dW + \frac{U}{2}\right) \sum_{x\in\Lambda} \hat{n}_x.$$

- U, W, μ: real parameters.
- The family of number operators $\{\hat{n}_{x\sigma}\}$ consists of commuting operators, thus $H_{\mathrm{C},\Lambda}$ is a classical Hamiltonian.
- The ground state of $H_{\mathrm{C},\Lambda}$ varies depending on the values of the parameters. $\rightsquigarrow$ Phase diagram

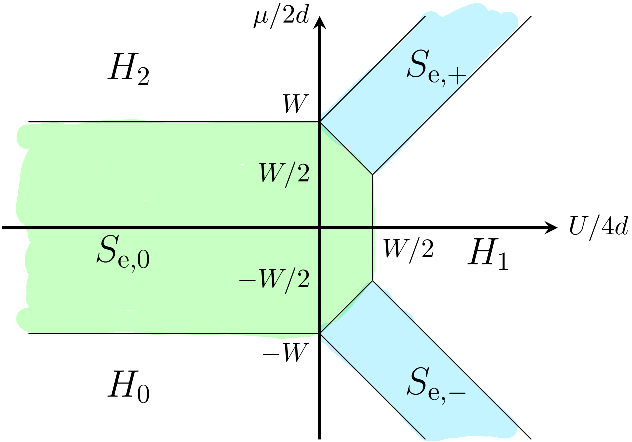


Figure: Phase diagram of the ground states for $H_{C,\Lambda}$.

Order parameter and phase diagram

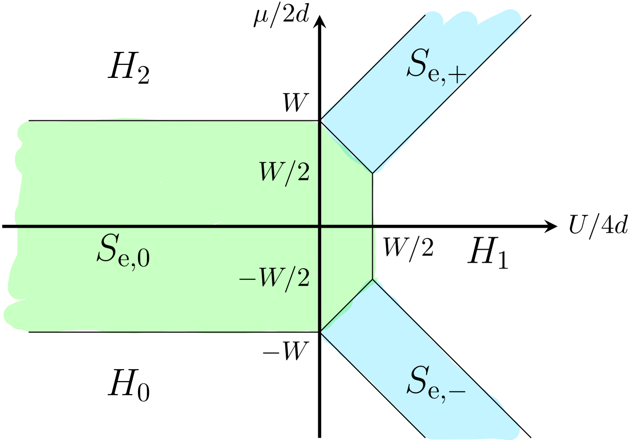
- Let $H_{\mathrm{C},\Lambda}^{(\mathrm{P})}$ be the Hamiltonian with the periodic boundary conditions imposed.
- $\mathfrak{A}_{\mathrm{e}}$ is the $*$-algebra generated by $\{c_{x,\sigma} : x \in \mathbb{Z}^d, \sigma \in \{\uparrow, \downarrow\}\}$.
- Given an observable $\Psi \in \mathfrak{A}_{\mathrm{e}}$, the thermal expectation value of Ψ with respect to $H_{\mathrm{C},\Lambda}^{(\mathrm{P})}$ is defined as

$$\langle \Psi \rangle_{\beta,\mathrm{C},\Lambda}^{(\mathrm{P})} = \frac{\mathrm{Tr}\left[\Psi \, \mathrm{e}^{-\beta H_{\mathrm{C},\Lambda}^{(\mathrm{P})}}\right]}{Z_{\mathrm{C},\Lambda}^{(\mathrm{P})}}, \quad Z_{\mathrm{C},\Lambda}^{(\mathrm{P})} = \mathrm{Tr}\left[\mathrm{e}^{-\beta H_{\mathrm{C},\Lambda}^{(\mathrm{P})}}\right].$$

- For any state $\langle \cdot \rangle$ on $\mathfrak{A}_{\mathrm{e}}$, we define the staggered density as

$$\Delta_{\mathrm{C}} = \lim_{L \to \infty} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} (-1)^x \langle \hat{n}_x \rangle.$$

When $\Delta_{\mathrm{C}} \neq 0$, a CDW emerges.



- $\Delta_{\mathrm{C}} = 1\,(S_{\mathrm{e},0})$, $\Delta_{\mathrm{C}} = 1/2\ (S_{\mathrm{e},\pm})$
- $\Delta_{\mathrm{C}} = 0\,(H_i,\ i = 0, 1, 2)$

CDW in itinerant electron systems

The extended Hubbard model:

$$H_{\mathrm{H},\Lambda} = T_\Lambda + H_{\mathrm{C},\Lambda}.$$

Here,

$$T_\Lambda = -t \sum_{\langle x,y\rangle \in E_\Lambda} \sum_{\sigma=\uparrow,\downarrow} (c_{x,\sigma}^* c_{y,\sigma} + c_{y,\sigma}^* c_{x,\sigma}).$$

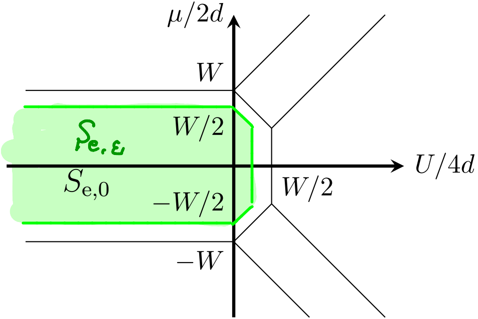
- Consider the electron hopping term as a quantum perturbation.
- It is logical to ask whether charge density waves are stable in the case $t \neq 0$.

Theorem 2.1 (Borgs and Kotecky(2000))

The charge density waves are stable, i.e., $\Delta_C \neq 0$ in the parameter region:

$$S_{e,\varepsilon} = \left\{ (U,\mu) \in \mathbb{R}^2 : U < 2d(W-\varepsilon),\ |\mu| < 2d \min\left\{ W-\varepsilon,\ W-\varepsilon-\frac{U}{4d} \right\} \right\},$$

if the temperature is low enough and $\varepsilon > 0$ and $|t|$ are sufficiently small.



- The proof is based on the Pirogov–Sinai theory.
  Remark: The method of reflection positivity can cover the case where $\mu = 0$ (half-filling system) only.
- The Pirogov–Sinai theory has emerged as a potent approach in classical statistical mechanics for delineating first-order phase transitions and coexistence of phases at low temperatures.[11]
- Noteworthy research works that utilize the Pirogov–Sinai theory to many-electron systems include Borgs-Kotecký-Ueltschi(1996), Datta-Fernández-Fröhlich(1996), Borgs-Kotecký(2000). [12]

[11] S. A. Pirogov and Y. G. Sinai, Theoretical and Mathematical Physics, 25(3):1185–1192, Dec. 1975.

[12] Borgs, C.; Kotecký, R.; Ueltschi, D. Comm. Math. Phys. 181 (1996), 409–446,
Borgs, C.; Kotecký, R. Comm. Math. Phys. 208 (2000), 575–604,
Datta, Nilanjana; Fernández, Roberto; Fröhlich, Jürg, J. Statist. Phys. 84 (1996), 455–534.

Part 3: Enhancement of Charge Density Waves by Electron-Phonon Interaction

The Holstein–Hubbard model

The Holstein–Hubbard model:

$$H_\Lambda = H_{\mathrm{H},\Lambda} + g \sum_{x\in\Lambda} \hat{n}_x (b_x + b_x^*) + \omega_0 \sum_{x\in\Lambda} b_x^* b_x .$$

- H_Λ acts on the following Hilbert space:

  $$\mathfrak{H}_\Lambda = \mathfrak{F}_\mathrm{e}(\Lambda) \otimes \mathfrak{F}_\mathrm{p}(\Lambda),$$

  where $\mathfrak{F}_\mathrm{p}(\Lambda)$ is the bosonic Fock space over $\ell^2(\Lambda)$:

  $$\mathfrak{F}_\mathrm{p}(\Lambda) = \bigoplus_{n=0}^{\infty} \otimes_\mathrm{s}^n \ell^2(\Lambda);$$

  $\otimes_\mathrm{s}^n \ell^2(\Lambda)$ stands for the n-fold symmetric tensor product of $\ell^2(\Lambda)$, with $\otimes_\mathrm{s}^0 \ell^2(\Lambda) = \mathbb{C}$.
- The annihilation- and creation operators of phonons are denoted by b_x and b_x^*, respectively.

  $$[b_x, b_y^*] = \delta_{x,y}, \quad [b_x, b_y] = 0.$$

- The phonons are assumed to be dispersionless with energy $\omega_0 > 0$.
- The parameter g is the strength of the electron-phonon interaction.
- Given $\varPsi \in \mathfrak{A}_{\mathrm{e}}$, the thermal expectation value of $\varPsi$ with respect to $H_\varLambda^{(\mathrm{P})}$ is defined as

$$\langle \varPsi \rangle_{\beta,\varLambda}^{(\mathrm{P})} = \frac{\mathrm{Tr}\left[\varPsi\, \mathrm{e}^{-\beta H_\varLambda^{(\mathrm{P})}}\right]}{Z_\varLambda^{(\mathrm{P})}}, \quad Z_\varLambda^{(\mathrm{P})} = \mathrm{Tr}\left[\mathrm{e}^{-\beta H_\varLambda^{(\mathrm{P})}}\right].$$

- As before, we define the staggered density as

$$\Delta = \lim_{L\to\infty} \frac{1}{|\varLambda|} \sum_{x\in\varLambda} (-1)^x \langle \hat{n}_x \rangle. \tag{1}$$

- We can prove the existence of charge density waves in the following region:

$$S_{\mathrm{ep},\varepsilon} = \left\{ (U,\mu) \in \mathbb{R}^2 : U < 2d(W-\varepsilon) + \frac{2g^2}{\omega_0}, \right.$$
$$\left. |\mu| < 2d \min\left\{ W-\varepsilon,\ W-\varepsilon - \frac{U}{4d} + \frac{g^2}{2d\omega_0} \right\} \right\},$$

provided that $|t|$ is small enough, ω_0 is large enough and at sufficiently low temperatures.
- $S_{\mathrm{e},\varepsilon} \subset S_{\mathrm{ep},\varepsilon} \rightsquigarrow$ The electron-phonon interaction has a significant stabilizing effect on the charge density waves.
- Such effects have been anticipated in theoretical physics based on numerical computations and discussions relying on certain approximation theories.

The spin operators, $(S_x^{(1)}, S_x^{(2)}, S_x^{(3)})$, at site x are defined to be

$$S_x^{(i)} = \frac{1}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} c_{x,\sigma}^* (s^{(i)})_{\sigma,\sigma'} c_{x,\sigma}, \quad i = 1, 2, 3,$$

where $s^{(i)}$ $(i = 1, 2, 3)$ are the Pauli matrices:

$$s^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ s^{(2)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ s^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$(s^{(i)})_{\sigma,\sigma'}$ represents the matrix elements of $s^{(i)}$, with the correspondence $\uparrow = 1, \downarrow = 2$[13].

[13]Under this convention, for example, $(s^{(1)})_{\uparrow,\uparrow} = (s^{(1)})_{1,1} = 0$ and $(s^{(1)})_{\uparrow,\downarrow} = (s^{(1)})_{1,2} = 1$.

Theorem 3.1 (T.M., 2023)

Suppose that $0 < \varepsilon < W$ *and* $(U, \mu) \in S_{\mathrm{ep},\varepsilon}$*. There exist certain constants* $0 < \beta_0 < \infty, 0 < \omega_* < \infty$ *and* $0 < t_0 < \infty$*, such that, if* $\beta \geq \beta_0, \omega_0 \geq \omega_*$ *and* $|t| \leq t_0$*, then the following* (i)-(iii) *hold:*

(i) *Given an arbitrary local observable* $\Psi \in \mathfrak{A}_{\mathrm{e}}$*, the infinite volume limit:*

$$\langle \Psi \rangle_\beta^{(\mathrm{P})} = \lim_{L\to\infty} \langle \Psi \rangle_{\beta,\Lambda}^{(\mathrm{P})}$$

converges. The state $\langle \cdot \rangle_\beta^{(\mathrm{P})}$ *on* $\mathfrak{A}_{\mathrm{e}}$ *defined in this way can be represented by the convex combination of two pure states:*

$$\langle \Psi \rangle_\beta^{(\mathrm{P})} = \frac{1}{2} \langle \Psi \rangle_\beta^{(+)} + \frac{1}{2} \langle \Psi \rangle_\beta^{(-)}.$$

The states $\langle \cdot \rangle_\beta^{(\pm)}$ *describe charge density waves:*

$$\langle \hat{n}_x \rangle_\beta^{(+)} = \rho + (-1)^x \Delta, \quad \langle \hat{n}_x \rangle_\beta^{(-)} = \rho - (-1)^x \Delta.$$

Here, $\Delta^{(+)} = -\Delta^{(-)} = \Delta > 0$*, where* $\Delta^{(+)}$ *and* $\Delta^{(-)}$ *are staggered densities defined with respect to states* $\langle \cdot \rangle_\beta^{(\pm)}$ *in equation* (1)*. Additionally,* ρ *is given as*

$$\rho = \lim_{L\to\infty} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \hat{n}_x \rangle_\beta^{(+)}$$

and coincides with the density associated with $\langle \cdot \rangle_\beta^{(-)}$*.*

Theorem 3.2 (Cont'd)

(ii) *No magnetic order exists. Namely,* $\langle S_x^{(i)} \rangle_\beta^{(\pm)} = 0$ $(i = 1, 2, 3)$ *holds for every* $x \in \mathbb{Z}^{\mathrm{d}}$.

(iii) *The two-point correlation function concerning arbitrary local observables* $\Psi, \Phi \in \mathfrak{A}_{\mathrm{e}}$ *decays exponentially. Namely, there are constants* $C_{\Psi,\Phi} > 0$ *and* $\xi_\ell > 0$ *such that*

$$\left| \langle \Psi\Phi \rangle_\beta^{(\pm)} - \langle \Psi \rangle_\beta^{(\pm)} \langle \Phi \rangle_\beta^{(\pm)} \right| \le C_{\Psi,\Phi} \exp\left\{ -\frac{\mathrm{dist}(\mathrm{supp}\,\Psi, \mathrm{supp}\,\Phi)}{\xi_\ell} \right\},$$

where, for any two finite subsets A *and* B *of* $\mathbb{Z}^{\mathrm{d}}$, *the distance between* A *and* B *is defined as* $\mathrm{dist}(A, B) = \min\{\|x - y\| : x \in A, y \in B\}$.

Outline of the proof

- HH Hamiltonian:

$$H_\Lambda = -t\sum_\sigma \sum_{\langle x;y\rangle} c^*_{x,\sigma} c_{y,\sigma} + H_{\mathrm{C},\Lambda} + g\sum_{x\in\Lambda} \hat{n}_x (b_x + b_x^*) + \omega_0 N_{\mathrm{p},\Lambda}$$

⇝ Lang–Firsov transformation:

$$H_\Lambda = -t\sum_\sigma \sum_{\langle x;y\rangle} \mathrm{e}^{\mathrm{i}\Phi_{x,y}} c^*_{x,\sigma} c_{y,\sigma} + H^{(\mathrm{eff})}_{\mathrm{C},\Lambda} + \omega_0 N_{\mathrm{p},\Lambda}$$

where

$$\Phi_{x,y} = -\frac{\sqrt{2}g}{\omega_0}(q_x - q_y), \quad q_x = \frac{1}{\sqrt{2}}\overline{(b_x + b_x^*)},$$

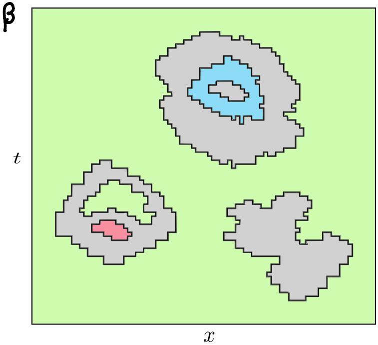
$$H^{(\mathrm{eff})}_{\mathrm{C},\Lambda} = U_{\mathrm{eff}} \sum_{x\in\Lambda} n_{x,\uparrow} n_{x,\downarrow} + W \sum_{\langle x,y\rangle} n_x n_y - \left(\mu + 2dW + \frac{U_{\mathrm{eff}}}{2}\right)\sum_{x\in\Lambda} n_x,$$

$$U_{\mathrm{eff}} = U - \frac{2g^2}{\omega_0}.$$

⇝ The partition function can be represented as a contour model on the $(d+1)$-dimensional spacetime $\mathbb{T}_\Lambda := [-L, L-1]^d \times [0, \beta]$:[14]

$$Z_{\ell,\Lambda} = \sum_{\{Y_1,\ldots,Y_n\}} \mathrm{e}^{-\tilde{\beta}\sum_\ell e_\ell |V_\ell|} \prod_{i=1}^{n} \rho(Y_i)$$

⇝ The Pirogov–Sinai theory can be applied.



[14] In the case of classical systems such as the Ising model, the partition function can be represented by a contour model in d-dimensional space.

Difficulties:

- The construction of the contour model is complex.
- Existing methods cannot be applied due to the non-conservation of the boson particle number.
  ⇝ It is challenging to control $\rho(Y)$.
  ⇝ By utilizing explicit formulas for correlation functions in quantum field theory, it becomes possible to evaluate $\rho(Y)$.

Theorem 3.3 (T.M. (2023))

Suppose that $0 < \varepsilon < W$ *and* $(U, \mu) \in S_{\mathrm{ep},\varepsilon}$. *There exist certain constants* $0 < \beta_0 < \infty, 0 < \omega_* < \infty$ *and* $0 < t_0 < \infty$, *such that, if* $\beta \geq \beta_0, \omega_0 \geq \omega_*$ *and* $|t| \leq t_0$, *then the following holds:*

$$|\rho(Y)| \leq \mathrm{e}^{-(\tilde{\beta} e_{\mathrm{e}} + \tilde{\beta} c + \gamma)|\mathrm{supp}\, Y|},$$

$$\left|\frac{\partial}{\partial \underline{\nu}_i} \rho(Y)\right| \leq \left(2\tilde{\beta} C_0 + \frac{\mathrm{e}}{\mathrm{e}-1} + \frac{5}{\alpha\, \mathrm{e}^{1/2}}\right) |\mathrm{supp}\, Y|\, \mathrm{e}^{-(\tilde{\beta} e_{\mathrm{e}} + \tilde{\beta} c + \gamma_\dagger)|\mathrm{supp}\, Y|},$$

where, $\gamma, \gamma_\dagger$ *and* c *are some positive numbers, and the symbols* $\underline{\nu}_i$ *represent the parameters* U, μ, W, g.

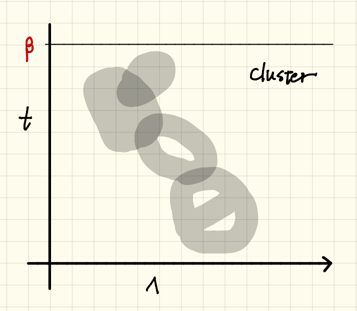
- Define the weight of the contour Y as follows:

$$W_\ell(Y) = \rho(Y)\, \mathrm{e}^{\tilde{\beta} e_\ell |Y|} \prod_{m=1}^{r} \frac{Z_{m,\mathrm{Int}\,_m Y}}{Z_{\ell,\mathrm{Int}\,_m Y}}.$$

⇝ The partition function can be expressed as follows:

$$Z_{\ell,\Lambda} = \mathrm{e}^{-\tilde{\beta} e_\ell |\Lambda|} \sum_{\{Y_1,\ldots,Y_n\} \subset \mathbb{T}_\Lambda^n} \prod_{i=1}^{n} W_\ell(Y_i).$$

- The set $X = \{Y_1, \ldots, Y_n\}$ is called a cluster, if $Y_1, \ldots, Y_n$ are "connected".



- For each cluster $X = \{Y_1, \dots, Y_n\}$, set

$$\Psi(X) = \left\{ \prod_{Y \in X} \frac{1}{n(Y)!} \right\} \varphi(Y_1, \dots, Y_n) \prod_{i=1}^{n} W_\ell(Y_i).$$

$\rightsquigarrow$ Cluster expansion:

$$\log Z_{\ell,\Lambda} = \sum_{X : \mathrm{supp}\, X \subset \Lambda} \Psi(X).$$

Proposition 3.4

$$f_\ell = - \lim_{L \to \infty} \frac{1}{|\Lambda|} \log Z_{\ell,\Lambda} = e_\ell - \frac{1}{\beta} \sum_{X : o \in \mathrm{supp}\, X} \frac{1}{|\mathrm{supp}\, X|} \Psi(X).$$

Proposition 3.5

The limit $\langle \Psi \rangle_\ell = \lim_{L\to\infty} \langle \Psi \rangle_{\ell,\Lambda}$ *exists, and the following holds:*

$$\langle \Psi \rangle_\ell = \sum_{\mathcal{Y}_\Psi} W_{\ell,\Psi}(\mathcal{Y}_\Psi) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{Y_1,\dots,Y_n\}} \left[\prod_{i=1}^{n} W_\ell(Y_i) \right] \varphi(\mathcal{Y}_\Psi, Y_1, \dots, Y_n).$$

Proposition 3.6

$$\langle \Psi\Phi \rangle_\ell - \langle \Psi \rangle_\ell \langle \Phi \rangle_\ell$$

$$= \sum_{\mathcal{Y}_\Psi, \mathcal{Y}_\Phi} W_{\ell,\Psi}(\mathcal{Y}_\Psi) W_{\ell,\Phi}(\mathcal{Y}_\Phi) \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \sum_{\{Y_3,\dots,Y_n\}} \left[\prod_{i=3}^{n} W_\ell(Y_i) \right] \varphi(\mathcal{Y}_\Psi, \mathcal{Y}_\Phi, Y_3, \dots, Y_n).$$

From this, it follows that :

$$|\langle \Psi\Phi \rangle_\ell - \langle \Psi \rangle_\ell \langle \Phi \rangle_\ell| \le C_{\Psi,\Phi} \exp\left\{ -\frac{\mathrm{dist}(\mathrm{supp}\, \Psi, \mathrm{supp}\, \Phi)}{\xi_\ell} \right\}.$$

Summary

- We provided a mathematical justification for the physicists' prediction that CDW stabilizes due to the electron-phonon interaction.
- The proof relied on the quantum Pirogov–Sinai theory.
- Consideration was limited to on-site electron-phonon interactions, but extension to finite range is possible (though the proof becomes more complex).
- There are many aspects of magnetic ordering that are not yet fully understood mathematically.

Thank you!