Low energy LAP for slowly decaying attractive potentials

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Introduction

Slowly decaying attractive potentials

We discuss the uniform resolvent estimates near zero energy for the Schrödinger operator

$$H = -\tfrac{1}{2}\Delta + V + q \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^d)$$

with $d \in \mathbb{N} = \{1, 2, \ldots\}$. The potential V is *slowly decaying* and *attractive*, and q is a short-range perturbation relative to V.

Slowly decaying attractive potentials

We use notation $\langle x\rangle = (1+x^2)^{1/2}$.

Conditions for V and q

Let $V \in C^2(\mathbb{R}^d)$ be spherically symmetric, and there exist $\nu, \epsilon \in (0,2)$ and $c, C > 0$ such that for any $|\alpha| \le 2$

$$|\partial^\alpha V(x)| \le C\langle x\rangle^{-\nu-|\alpha|}, \quad V(x) \le -c\langle x\rangle^{-\nu}, \quad x\cdot(\nabla V(x)) \le -(2-\epsilon)V(x).$$

In addition, let $q \in L^\infty(\mathbb{R}^d)$, and there exist $\nu' \in (\nu, 2]$ and $C' > 0$ such that

$$|q(x)| \le C'\langle x\rangle^{-1-\nu'/2}.$$

Remark (Aim of this talk)

Nakamura' 94, Fournais-Skibsted' 04, Richard' 06 and Skibsted' 13 discussed the LAP for C^∞ slowly decaying attractive potentials. We extend previous results to the C^2 potentials in the framework of an appropriate Agmon-Hörmander spaces.

Agmon-Hörmander spaces

We introduce the *Agmon-Hörmander spaces*. We set $r(x) = |x|$ and define an *effective time* as

$$\tau(\lambda, x) = \int_0^r a(\lambda, s)^{-1}\, \mathrm{d}s; \quad a(\lambda, r) = (2 \max\{\lambda, 0\} - 2V(r))^{1/2}.$$

The function τ means the time of arrival at distance r from the origin for classical orbit with energy λ starting at $r = 0$ at time $t = 0$.
Moreover, we note that the following estimates hold for τ:

$$cr\langle r\rangle^{\nu/2} \leq \tau(0, x) \leq Cr\langle r\rangle^{\nu/2},$$

$$c'\langle\lambda\rangle^{-1/2} r \leq \tau(\lambda, x) \leq C'\langle\lambda\rangle^{-1/2} r \ \text{ uniformly in } \lambda \geq \lambda_0 > 0.$$

From these estimates, we can understand that the scattering velocity of particles corresponding to zero energy is slower than that of particles corresponding to positive energy.

Agmon-Hörmander spaces

We define the *Agmon-Hörmander spaces* associated with τ as

$$\mathcal{B}(\lambda) = \{\psi \in L^2_{\mathrm{loc}};\ \|\psi\|_{\mathcal{B}(\lambda)} < \infty\}, \quad \|\psi\|_{\mathcal{B}(\lambda)} = \sum_{n\in\mathbb{N}} 2^{n/2}\|1_n(\lambda)\psi\|_{\mathcal{H}},$$

$$\mathcal{B}^*(\lambda) = \{\psi \in L^2_{\mathrm{loc}};\ \|\psi\|_{\mathcal{B}^*(\lambda)} < \infty\}, \quad \|\psi\|_{\mathcal{B}^*(\lambda)} = \sup_{n\in\mathbb{N}} 2^{-n/2}\|1_n(\lambda)\psi\|_{\mathcal{H}},$$

$$\mathcal{B}^*_0(\lambda) = \Big\{\psi \in \mathcal{B}^*(\lambda);\ \lim_{n\to\infty} 2^{-n/2}\|1_n(\lambda)\psi\|_{\mathcal{H}} = 0\Big\},$$

where we let

$$1_1(\lambda) = 1\big(\{x \in \mathbb{R}^d;\ \tau(\lambda, x) < 2\}\big),$$
$$1_n(\lambda) = 1\big(\{x \in \mathbb{R}^d;\ 2^{n-1} \le \tau(\lambda, x) < 2^n\}\big) \ \text{ for } n = 2, 3, \ldots$$

with $1(S)$ being the characteristic function of a subset $S \subset \mathbb{R}^d$.

Agmon-Hörmander spaces

For any $\lambda \in \mathbb{R}$, $s > 1/2$, the following relations for weighted L^2 spaces and Agmon-Hörmander spaces hold

$$L^2_{s,\lambda} \subsetneq \mathcal{B}(\lambda) \subsetneq L^2_{1/2,\lambda} \subsetneq \mathcal{H} \subsetneq L^2_{-1/2,\lambda} \subsetneq \mathcal{B}^*_0(\lambda) \subsetneq \mathcal{B}^*(\lambda) \subsetneq L^2_{-s,\lambda},$$

where

$$L^2_{s,\lambda} = \langle \tau \rangle^{-s} \mathcal{H} \quad \text{for } s \in \mathbb{R}.$$

The above relationship can be understood from the following inequality.
<u>Recall</u>

$$cr\langle r \rangle^{\nu/2} \leq \tau(0,x) \leq Cr\langle r \rangle^{\nu/2},$$

$$c'\langle \lambda \rangle^{-1/2} r \leq \tau(\lambda,x) \leq C'\langle \lambda \rangle^{-1/2} r \quad \text{uniformly in } \lambda \geq \lambda_0 > 0.$$

Main results

Rellich's theorem

First, we introduce Rellich's theorem. This theorem is an important basis for our approach.

Theorem 1 (Rellich's theorem)

If $\phi \in \mathcal{B}_0^*(\lambda)$ *with* $\lambda \geq 0$ *satisfies*

$$(H - \lambda)\phi = 0 \quad \textit{in the distributional sense,}$$

then $\phi \equiv 0$. *In particular, the self-adjoint realization of* H *on* $\mathcal{H}$ *does not have non-negative eigenvalues.*

Remark

It is known that Zero is always the accumulation point of negative eigenvalues.

LAP bounds

Next come the *LAP bounds* for the resolvent

$$R(z) = (H - z)^{-1} \in \mathcal{L}(\mathcal{H}) \ \text{ for } z \in \rho(H).$$

We define the region where the spectral parameter varies as follows:

$$\Gamma_\pm(\rho, \omega) = \{ z \in \mathbb{C};\ 0 < |z| < \rho,\ 0 < \pm \arg z < \omega \}$$

for any $\rho > 0$ and $\omega \in (0, \pi)$.

Theorem 2 (LAP bounds)

There exists $C > 0$ such that for any $z = \lambda \pm \mathrm{i}\mu \in \Gamma_\pm(\rho, \omega)$ and $\psi \in \mathcal{B}(\lambda)$

$$\|R(z)\psi\|_{\mathcal{B}^*(\lambda)} \leq C \|\psi\|_{\mathcal{B}(\lambda)}.$$

In particular, the self-adjoint realization of H on $\mathcal{H}$ does not have the singular continuous spectrum on $[0, \infty)$.

Radiation condition bounds and their applications

We set an asymptotic complex phase b:

$$b = b_{\pm z} = \sqrt{2(z - V)} \mp \mathrm{i}\frac{\partial_r V}{4(z - V)} \quad \text{for} \quad z \in \mathbb{C}\backslash(-\infty, 0]$$

respectively, where ∂_r is the radial differential operator.
We also set for any $\rho > 0$

$$\beta_{c,\rho} = \min\left\{\frac{2-\nu}{2(2+\nu)}, \frac{\nu' - \nu}{2+\nu}, \frac{2+3\epsilon}{8}\liminf_{|x|\to\infty}\left(\inf_{\lambda\in[0,\rho]}\tau a r^{-1}\right)\right\} > 0.$$

Next, we introduce the radiation condition bounds.

Theorem 3 (Radiation condition bounds for $R(z)$)

For all $\beta \in [0, \beta_{c,\rho})$ *there exists* $C > 0$ *such that for any* $z = \lambda \pm \mathrm{i}\mu \in \Gamma_\pm(\rho, \omega)$ *and* $\psi \in \langle\tau\rangle^{-\beta}\mathcal{B}(\lambda)$

$$\|\tau^\beta a^{-1}(p_r \mp b_z)R(z)\psi\|_{\mathcal{B}^*(\lambda)} \le C\|\langle\tau\rangle^\beta\psi\|_{\mathcal{B}(\lambda)},$$

where $p_r = (-\mathrm{i})\partial_r$.

Radiation condition bounds and their applications

By combining the LAP bounds and the radiation condition bounds, we obtain the following theorem.

Corollary 4 (LAP)

For any $s > 1/2$ *and* $\gamma \in (0, \min\{s - 1/2, \beta_{c,\rho}\})$ *there exists* $C > 0$ *such that for* $z, z' \in \Gamma_+(\rho, \omega)$ *or* $z, z' \in \Gamma_-(\rho, \omega)$

$$\|R(z) - R(z')\|_{\mathcal{B}\left(L^2_{s,\min\{\mathrm{Re}\, z, \mathrm{Re}\, z'\}}, L^2_{-s,\min\{\mathrm{Re}\, z, \mathrm{Re}\, z'\}}\right)} \leq C|z - z'|^\gamma.$$

The operator $R(z)$ *attains uniform limits as* $z \in \Gamma_\pm(\rho, \omega) \to \lambda \in [0, \rho)$ *in* $\mathcal{B}(L^2_{s,\lambda}, L^2_{-s,\lambda})$:

$$R(\lambda \pm \mathrm{i}0) = \lim_{z \in \Gamma_\pm(\rho,\omega) \to \lambda} R(z).$$

These limit $R(\lambda \pm \mathrm{i}0)$ *belongs to* $\mathcal{B}(\mathcal{B}(\lambda), \mathcal{B}^*(\lambda))$.

Now we have the limiting resolvents $R(\lambda \pm \mathrm{i}0)$, and the radiation condition bounds extend to them.

Corollary 5 (Radiation condition bounds for $R(\lambda \pm \mathrm{i}0)$)

For all $\beta \in [0, \beta_{c,\rho})$ *there exists* $C > 0$ *such that for any* $\lambda \in [0, \rho)$, $\psi \in \langle \tau \rangle^{-\beta} \mathcal{B}(\lambda)$

$$\|\tau^{\beta} a^{-1} (p_r \mp b_{\lambda}) R(\lambda \pm \mathrm{i}0) \psi\|_{\mathcal{B}^*(\lambda)} \leq C \|\langle \tau \rangle^{\beta} \psi\|_{\mathcal{B}(\lambda)}.$$

Finally we characterize the limiting resolvents $R(\lambda \pm \mathrm{i}0)$.

Corollary 6 (Sommerfeld's uniqueness theorem)

Let $\lambda \geq 0$, $\phi \in L^2_{\mathrm{loc}}$ *and* $\psi \in \langle\tau\rangle^{-\beta}\mathcal{B}(\lambda)$ *with* $\beta \in [0, \beta_{c,\lambda})$ *Then* $\phi = R(\lambda \pm \mathrm{i}0)\psi$ *holds iff both of the following conditions hold:*

(i) $(H - \lambda)\phi = \psi$ *in the distributional sense.*

(ii) $\phi \in \langle\tau\rangle^{\beta}\mathcal{B}^*(\lambda)$ *and* $a^{-1}(p_r \mp b_\lambda)\phi \in \langle\tau\rangle^{-\beta}\mathcal{B}^*_0(\lambda)$.

Classical mechanics

Classical mechanics

We discuss the classical mechanics for the classical Hamiltonian

$$H(x,p) = \tfrac{1}{2}p^2 + V(x).$$

Here we try to understand rolls of the effective time.

Proposition

If $(x(t), p(t))$ is a classical orbit of nonnegative energy $\lambda \geq 0$ with positive radial momentum at $t = 0$;

$$p_r(0) := |x(0)|^{-1} x(0) \cdot p(0) > 0,$$

then for any $t \geq 0$

$$\tau(\lambda, x(t)) \geq a(\lambda, x(0))^{-1} p_r(0) t + \tau(\lambda, x(0)).$$

From this Proposition, it is indicated that the physical quantity τ is increasing with respect to t as time passes.

Effective time

Proof of Proposition

It suffices to show that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tau(\lambda, x(t)) \geq 0.$$

Below we would like to avoid explicit t-derivatives to motivate the later stationary approach to the quantum mechanics. For that we employ the Poisson brackets. Let

$$D = \frac{\mathrm{d}}{\mathrm{d}t} = \{H, \cdot\},$$

and introduce a classical observable A as

$$A = D\tau = \{H, \tau\} = a^{-1}p_r.$$

Effective time

We compute

$$D^2\tau = \{H, A\} = a^{-3}(\partial_r V)p_r^2 + a^{-1}p \cdot (\nabla^2 r)p - a^{-1}(\partial_r V).$$

Using Condition for V, the law of conservation of energy and the convexity $\nabla^2 r \geq 0$, we can proceed as

$$\begin{aligned} D^2\tau &\geq a^{-3}\big[(\partial_r V)p_r^2 + (\partial_r V)p \cdot (r\nabla^2 r)p + (2\lambda - \epsilon V)p \cdot (\nabla^2 r)p - a^2(\partial_r V)\big] \\ &= a^{-3}\big[(2\lambda - \epsilon V)p \cdot (\nabla^2 r)p + 2(\partial_r V)(H - \lambda)\big] \\ &\geq 0. \end{aligned}$$

This completes the proof.
To prove our main result, we evaluate the quantization of the Poisson bracket between H and A.

Outline of proof

Outline of proof of LAP bounds

We only sketch the proof of LAP bounds. The proof depends on a commutator argument.

Theorem (LAP bounds)

There exists $C > 0$ such that for any $z \in \Gamma_{\pm}(\rho, \omega)$ and $\psi \in \mathcal{B}(\lambda)$

$$\|R(z)\psi\|_{\mathcal{B}^*(\lambda)} \leq C\|\psi\|_{\mathcal{B}(\lambda)}.$$

Let us introduce

$$A = \mathrm{i}[H, \tau] = \mathrm{Re}(a^{-1}p_r); \quad p_r = -\mathrm{i}\partial_r.$$

We also choose a cut off function $\chi \in C_0^\infty(\mathbb{R}^d)$ with support in $\{\tau \geq 1\}$.

Outline of proof of LAP bounds

We use a weight function of the form

$$\Theta = \chi\theta,$$

where

$$\theta = \int_0^{\tau/R} (1+s)^{-1-\delta}\,\mathrm{d}s = \delta^{-1}\big[1 - (1+\tau/R)^{-\delta}\big]; \quad \delta > 0, \quad R \geq 1.$$

We are going to compute and bound a *distorted commutator*

$$\mathrm{Im}(A\Theta(H-z)),$$

and we obtain a key estimate to prove LAP bounds. Thus the LAP bounds holds.

Thank you for your attention