Semiclassical limit of orthonormal Strichartz estimates on scattering manifolds

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Aim of this talk

- To see some effects of geometry on long-time behavior of solutions to the Schrödinger equation

- To investigate quantum-classical correspondence for the Strichartz estimates

Strichartz estimates

Let P be a self-adjoint operator on $L^2(X, dg)$, where (X, g) is a (noncompact) Riemannian manifold. The Strichartz estimates for the Schrödinger equation:

$$\begin{cases} i\partial_t u(t) = Pu(t) \\ u(0) = u_0. \end{cases}$$

are inequalities like

$$\|e^{-itP}u_0\|_{L_t^q L_z^r} \lesssim \|u_0\|_2,$$

where $\|\cdot\|_p = \|\cdot\|_{L^p(X,dg)}$ and $L_t^q L_z^r = L^q(\mathbb{R}_t; L^r(X, dg(z)))$. If $r > 2$ these estimates represent smoothing effects by the propagator.

Strichartz estimates for the free Laplacian

Example (Strichartz '77, Ginibre-Velo '85, Yajima '87, Keel-Tao '98)

For the free Laplacian on $L^2(\mathbb{R}^d)$, $d \geq 1$, the Strichartz estimates:

$$\|e^{it\Delta}u_0\|_{L_t^q L_z^r} \lesssim \|u_0\|_2 \tag{0.1}$$

hold if and only if (q, r) satisfies the admissible condition:

$$(q, r) \in [2, \infty]^2, \quad \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \quad (d, q, r) \neq (2, 2, \infty).$$

(0.1) is equivalent to the validity of

$$\left\| \sum_{n=0}^{\infty} \nu_n |e^{it\Delta} f_n|^2 \right\|_{L_t^{\frac{q}{2}} L_z^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^1}$$

for all orthonormal systems $\{f_n\} \subset L^2(\mathbb{R}^d)$ and all complex-valued sequences $\nu = \{\nu_n\}$.

Strichartz estimates for orthonormal functions

Recently the orthonormal Strichartz estimates (Strichartz estimates for orthonormal functions) are proved for the free Laplacian on $L^2(\mathbb{R}^d)$.

Theorem (Frank-Lewin-Lieb-Seiringer '14, Frank-Sabin '17)

Assume either of the following:

1. $d \geq 1, q, r \in [2, \infty], \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), r \in [2, \frac{2(d+1)}{d-1})$ *and* $\beta = \frac{2r}{r+2}$.
2. $d \geq 3, q, r \in [2, \infty], \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), r \in [\frac{2(d+1)}{d-1}, \frac{2d}{d-2})$ *and* $2\beta < q$.

Then

$$\left\| \sum_{n=0}^{\infty} \nu_n |e^{it\Delta} f_n|^2 \right\|_{L_t^{\frac{q}{2}} L_z^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta}$$

holds for all orthonormal systems $\{f_n\}$ *and all complex-valued sequences* $\nu = \{\nu_n\}$.

Since $\beta \geq 1$ for all (q, r) and $\beta > 1$ if $r > 2$, these are better than the ordinary Strichartz estimates. The range of β in (1) is optimal.

Remark 1 (density of operator)

For all orthonormal systems $\{f_n\} \subset L^2(X, dg)$ and all $\nu = \{\nu_n\} \in \ell^\beta$, $\beta \in [1, \infty)$, we set $\gamma := \sum_{n=0}^\infty \nu_n |f_n\rangle\langle f_n| \in \mathcal{L}(L^2)$ $(|f\rangle\langle f|g := \langle f, g\rangle f)$. Then

$$\rho(e^{-itP}\gamma e^{itP}) = \sum_{n=0}^{\infty} \nu_n |e^{-itP} f_n|^2.$$

· For $A \in \mathcal{L}(L^2)$, $\rho(A)$: the density of A is formally defined by $\rho(A)(z) = K_A(z, z)$, where $K_A(z, z')$ is the distribution kernel of A.

· $\gamma(t) = e^{-itP}\gamma e^{itP}$ is a solution to the Heisenberg equation:

$$\begin{cases} i\partial_t \gamma(t) = [P, \gamma(t)] \\ \gamma(0) = \gamma. \end{cases}$$

· Total number of particles $= \mathrm{Tr}(\gamma(t)) = \mathrm{Tr}(\gamma)$.

Remark 2 (Why restricted to the diagonal ?)

To see smoothing properties of $\gamma(t)$, restriction to $\mathrm{diag} = \{(z,z) \mid z \in X\}$ is natural. Consider $(X,g) = (\mathbb{R}^d, g)$, g: dz^2+ perturbation, $P = -\Delta_g$. If $\gamma = a^w(z, D_z)$,

$$\gamma(t) = e^{-itP} a^w(z, D_z) e^{itP} \sim b_t^w(z, D_z) =: B_t$$

with $b_t(z,\zeta) = a \circ e^{-tH_p}(z,\zeta)$, $p(z,\zeta) = g^{ij}(z)\zeta_i\zeta_j$ and $H_p = \frac{\partial p}{\partial \zeta}\frac{\partial}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial}{\partial \zeta}$. Since

$$WF(K_{B_t}) \subset N^* \mathrm{diag} \backslash 0 = \{(z,\zeta,z,-\zeta) \mid \zeta \neq 0\},$$

K_{B_t} is smooth outside diag.

Notations: $(g^{ij}) = (g_{ij})^{-1}, g = g_{ij} dz^i dz^j$.

$$a^w(z, D_z)u(z) = \int e^{i(z-z')\zeta} a(\tfrac{z+z'}{2}, \zeta) u(z') dz' d\zeta.$$

Setting

We consider scattering manifolds (manifolds with asymptotically conic ends).

Assumption (A)

(M°, g) is a d-dimensional noncompact complete Riemannian manifold with $d \geq 3$. There exists a compact subset $K \subset M^\circ$ such that $M^\circ \backslash K$ is diffeomorphic to $(0, \infty) \times Y$. Here Y is a $(d-1)$-dimensional compact connected manifold. We also assume that there exists a compactification M of M° such that $\partial M = Y$ and in a collar neighborhood of ∂M, $[0, \epsilon_0)_x \times Y_y$, g takes a form $g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}$. Here $h \in C^\infty([0, \epsilon_0); S^2 T^* Y)$.

Assumption (B)

$V \in C^\infty(M)$ satisfies $V(x, y) = \mathcal{O}(x^{2+\epsilon})$ near ∂M for some $\epsilon > 0$.

Known results

Definition

(M°, g) is nontrapping if every geodesic $z : \mathbb{R} \to M^\circ$ goes to ∂M as $t \to \pm\infty$.

- Local smoothing estimates $\|e^{it\Delta_g} u_0\|_{L^2_{loc} H^{\frac{1}{2}}_{loc}} \lesssim \|u_0\|_2$: Doi '96 (nontrapping), Nonnenmacher-Zworski '09 (logarithmic derivative loss by mild-trapping)
- Strichartz estimates: Hassell-Zhang '16 (nontrapping), Zhang-Zheng '17, Bouclet-Mizutani '24 (mild-trapping), Burq-Guillarmou-Hassell '10 (no global-in-time estimate with an elliptic stable periodic trajectory)
- Smoothness of fundamental solution: Doi '00 (nontrapping), Taira '23 (mild-trapping, non-smoothness with an elliptic periodic geodesic)

Remark

All negative results depend on quasimodes concentrating on a periodic trajectory.

Main results

Theorem

Let (M°, g) and V be as in Assumptions A and B. Suppose (M°, g) is nontrapping. We also assume the Schrödinger operator $P = -\Delta_g + V$ has neither nonpositive eigenvalues nor zero resonances. Then

$$\left\| \sum_{j=0}^{\infty} \nu_j |e^{-itP} f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_z^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta}$$

hold for any orthonormal system $\{f_j\} \subset L^2(M^\circ, dg)$ and any complex-valued sequence $\nu = \{\nu_j\}$. Here admissible pair (q, r) and $\beta \in [1, \infty]$ satisfy either of the following conditions: If $r \in [2, \frac{2(d+1)}{d-1})$, then $\beta = \frac{2r}{r+2}$, and if $r \in [\frac{2(d+1)}{d-1}, \frac{2d}{d-2})$, then $\beta < \frac{q}{2}$.

The important part is $r \in [2, \frac{2(d+1)}{d-1})$ since the other part follows from interpolation.

Remark (conjugate points)

Definition

Let $z : [0, t_0] \to M^\circ$ be a geodesic. $z(t_0)$ is a conjugate point of $z(0)$ if and only if $\exp_{z(0)}$ is singular at $t_0\dot{z}(0)$.

If $(z, z') \in M^\circ \times M^\circ$ is a conjugate point, the dispersive estimate

$$|e^{it\Delta_g}(z, z')| \lesssim |t|^{-\frac{d}{2}}$$

fails (Hassell-Wunsch '05). The orthonormal Strichartz estimates hold since we only need pointwise estimates microlocalized near the diagonal:

$$|U(t)U(s)^*(z, z')| \lesssim |t|^{-\frac{d}{2}}.$$

Roughly $U(t) = Qe^{it\Delta_g}$ with some ΨDO Q supported in a sufficiently small region.

Quantum and classical correspondense

We use the scattering symbol class

$$S^{k,l} := S\left(\langle\zeta\rangle^k\langle z\rangle^l, \frac{dz^2}{\langle z\rangle^2} + \frac{d\zeta^2}{\langle\zeta\rangle^2}\right) = \{a \in C^\infty(T^*\mathbb{R}^d) \mid |\partial_z^\alpha\partial_\zeta^\beta a(z,\zeta)| \lesssim \langle\zeta\rangle^{k-|\beta|}\langle z\rangle^{l-|\alpha|}\}$$

since the compactness of operators is important. By the asymptotic formula $[a^w(z,hD_z), b^w(z,hD_z)] \sim \frac{h}{i}\{a,b\}^w(z,hD_z)$, the following equations correspond:

$$\begin{cases} ih\partial_t\gamma(t) = [-h^2\Delta_g, \gamma(t)] \\ \gamma(0) = \gamma \end{cases} \qquad (\gamma(t) = e^{-ith\Delta_g}\gamma e^{ith\Delta_g})$$

$$\begin{cases} \partial_t f(t,z,\zeta) + H_p f(t,z,\zeta) = 0 \\ f(0,z,\zeta) = f_0(z,\zeta) \end{cases} \qquad (f(t,z,\zeta) = f_0 \circ e^{-tH_p}(z,\zeta))$$

where $p(z,\zeta) = g^{ij}(z)\zeta_i\zeta_j$.

Main results

Theorem

Assume $p \in S^{2,0}$ is real-valued, homogeneous of degree 2, H_p is complete on $T^*\mathbb{R}^d$ and $P = p^w(z, D_z)$ is essentially self-adjoint on $L^2(\mathbb{R}^d)$ with its core $C_0^\infty(\mathbb{R}^d)$. If

$$\left\| \sum_{j=0}^{\infty} \nu_j |e^{-itP} f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_z^{\frac{r}{2}}} \lesssim \|\nu\|_{\ell^\beta} \tag{0.2}$$

holds for any orthonormal system $\{f_j\}$ in $L^2(\mathbb{R}^d)$, complex-valued sequences $\nu = \{\nu_j\}$ and some (q, r, β) satisfying $q \in [2, \infty]$, $r \in [2, \infty)$, $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{2r}{r+2}$, then

$$\left\| \int_{\mathbb{R}^d} f \circ e^{-tH_p}(z, \zeta) d\zeta \right\|_{L_t^{\frac{q}{2}} L_z^{\frac{r}{2}}} \lesssim \|f\|_{L^\beta_{z,\zeta}} \tag{0.3}$$

holds for the same (q, r, β).

Main results

Remark

We don't need geometric assumptions except for the completeness of H_p and the essential self-adjointness of P, for example, nontrapping conditions or absence of conjugate points.

The following quadruplet naturally appears in the Strichartz estimates for the kinetic transport equations.

Definition

$(q, r, p, a) \in [1, \infty]^4$ is called a KT-admissible quadruplet if and only if $a = \mathrm{HM}(p, r)$, $\frac{1}{q} = \frac{d}{2}(\frac{1}{p} - \frac{1}{r})$, $p_*(a) \leq p \leq a \leq r \leq r_*(a)$ and $(q, r, p, d) \neq (a, \infty, \frac{a}{2}, 1)$ hold. Here $\mathrm{HM}(p, r)$ is the harmonic mean of p and r, i.e. $\mathrm{HM}(p, r)^{-1} = \frac{1}{2}(\frac{1}{p} + \frac{1}{r})$. If $\frac{d+1}{d} \leq a \leq \infty$, then $(p_*(a), r_*(a)) = (\frac{da}{d+1}, \frac{da}{d-1})$. If $1 \leq a \leq \frac{d+1}{d}$, then $(p_*(a), r_*(a)) = (1, \frac{a}{2-a})$. A KT-admissible quadruplet (q, r, p, a) is called the endpoint if $(q, r, p) = (a, r_*(a), p_*(a))$ and $\frac{d+1}{d} \leq a < \infty$ hold.

Main results

Corollary (Strichartz estimates for transport equations)

Let $p \in S^{2,0}$ be as in the above theorem. Suppose (0.2) holds for any (q, r, β) satisfying $q, r \in [2, \infty]$, $r \in [2, \frac{2(d+1)}{d-1})$, $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{2r}{r+2}$. Then

$$\|f \circ e^{-tH_p}\|_{L_t^q L_z^r L_\zeta^p} \lesssim \|f\|_{L^a_{z,\zeta}} \tag{0.4}$$

holds for any non-endpoint KT-admissible quadruplet (q, r, p, a). Furthermore If (q, r, p, a) and $(\tilde{q}, \tilde{r}, \tilde{p}, a')$ are non-endpoint KT-admissible quadruplet, then

$$\left\| \int_0^t F(s) \circ e^{-(t-s)H_p}(z, \zeta) ds \right\|_{L_t^q L_z^r L_\zeta^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_z^{\tilde{r}'} L_\zeta^{\tilde{p}'}} \tag{0.5}$$

holds.

Example and known results

Example

Let g be a nontrapping scattering metric on $\mathbb{R}^d$ ($d \geq 3$), which satisfies $|\partial_z^\alpha g_{ij}(z)| \lesssim \langle z\rangle^{-|\alpha|}$ for any $1 \leq i,j \leq d$ and $\alpha \in \mathbb{N}_0^d$. Then the assumptions in the above corollary are satisfied for $p(z,\zeta) = \sum_{i,j=1}^d g^{ij}(z)\zeta_i\zeta_j \in S^{2,0}$.

Remark

Consider $p(z,\zeta) = |\zeta|^2$. At the endpoint, (0.4) fails (Bennett-Bez-Gutierrez-Lee '14). (0.4) holds if and only if (q,r,p,a) is a non-endpoint KT-admissible quadruplet.

- $p(z,\zeta) = |\zeta|^2$: Castella-Perthame '96, Keel-Tao '98, Ovcharov '11
- 1D and $p(z,\zeta) = g(z)\zeta^2$, $g \sim 1$: weighted estimates by Salort '06
- nontrapping compactly supported perturbation of dz^2: Salort '07
- nontrapping long-range perturbation of dz^2: Salort '07 (with derivative loss)

Counterexamples of orthonormal Strichartz estimates

Definition

We say that the sharp orthonormal Strichartz estimates fail if and only if for any (q, r, β) satisfying $q, r \in [2, \infty)$, $r \in [2, \frac{2(d+1)}{d-1})$, $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{2r}{r+2}$, (0.2) does not hold uniformly in orthonormal $\{f_j\} \subset L^2$ and $\nu = \{\nu_j\}$.

Let $p \in S^{2,0}$ is real-valued, homogeneous of degree 2, H_p is complete on $T^*\mathbb{R}^d$ and $P = p^w(z, D_z)$ is essentially self-adjoint on $L^2(\mathbb{R}^d)$.

Corollary

Assume $d = 1$. If there exists a periodic trajectory $\gamma \subset T^*\mathbb{R}$ associated to H_p, the sharp orthonormal Strichartz estimates fail.

Corollary

If there exists a periodic stable trajectory $\gamma \subset T^*\mathbb{R}^d$ associated to H_p, the sharp orthonormal Strichartz estimates fail.

Example

Let N be the north pole of $\mathbb{S}^d$ for $d \geq 2$. We define $F : \mathbb{S}^d \setminus \{N\} (\subset \mathbb{R}^{d+1}) \to \mathbb{R}^d$ by $F(z_1, \ldots, z_{d+1}) = (\frac{z_1}{1-z_{d+1}}, \ldots, \frac{z_d}{1-z_{d+1}})$ and $G : \mathbb{R}^d \to \mathbb{B}^d$ by $G(z) = \frac{z}{\langle z \rangle}$. For r_0, $\epsilon \in (0,1)$ satisfying $r_0 + 2\epsilon < 1$, we take a cutoff function $\chi \in C^\infty([0,\infty);[0,1])$ such that

$$\chi(r) = \begin{cases} 1 & (r \leq r_0 + \epsilon) \\ 0 & (r \geq r_0 + 2\epsilon) \end{cases}$$

Example

If r_0 and ϵ are sufficiently small, then

$$g_{sc} := (1 - \chi(|z|))dz^2 + \chi(|z|)(G^{-1})^*(F^{-1})^* g_{\mathbb{S}^d \setminus \{N\}}$$

is a well-defined scattering metric on $\mathbb{R}^d$. Furthermore the sharp orthonormal Strichartz estimates fail for $\Delta_{g_{sc}}$.

Thank you for your attention !

Application to Boltzmann equation on scattering manifolds

Consider the Boltzmann equation on $T^*\mathbb{R}^d$.

$$\begin{cases} \partial_t f(t,z,\zeta) + H_p f(t,z,\zeta) = Q(f,f)(t,z,\zeta) \\ f(0,z,\zeta) = f_0(z,\zeta) \end{cases} \tag{B}$$

The nonlinearity in the right hand side of (B) is given by

$$Q(f,f)(t,z,\zeta) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} (f' f'_* - f f_*) B(\zeta - \zeta_*, \omega) d\omega d\zeta_*,$$

where $f' = f(t,z,\zeta')$, $f'_* = f(t,z,\zeta'_*)$, $f_* = f(t,z,\zeta_*)$ and the relations of pre-collisional and post-collisional momentum are $\zeta' = \zeta - [\omega\cdot(\zeta-\zeta_*)]\omega$, $\zeta'_* = \zeta_* + [\omega\cdot(\zeta-\zeta_*)]\omega$. We consider the inverse power law model $B(\zeta-\zeta_*,\omega) = |\zeta-\zeta_*|^{-1} b(\cos\theta)$, $\cos\theta = \frac{(\zeta-\zeta_*)\cdot\omega}{|\zeta-\zeta_*|}$ with cut-off condition $0 \le \int_{\mathbb{S}^{d-1}} b(\cos\theta) d\omega < \infty$ (high temperature).
Set $\Lambda = \left\{ (q,r,p) \in [1,\infty]^3 \mid \frac{1}{q} = \frac{d}{p} - 1, \frac{1}{r} = \frac{2}{d} - \frac{1}{p}, \frac{1}{d} < \frac{1}{p} < \frac{d+1}{d^2} \right\}$.

Application to Boltzmann equation on scattering manifolds

Theorem

Assume $d = 3$, (0.4) and (0.5) for any non-endpoint KT-admissible quadruplet (q, r, p, a) and $(\tilde{q}, \tilde{r}, \tilde{p}, a')$. If $f_0 \in L^3 \cap L^{\frac{15}{8}}_{z,\zeta}$ satisfies $f_0 \geq 0$ and $\|f_0\|_{L^3 \cap L^{\frac{15}{8}}_{z,\zeta}}$ is sufficiently small, then (B) has a unique nonnegative solution $f \in C([0,\infty); L^3_{z,\zeta}) \cap L^{\boldsymbol{q}}([0,\infty); L^{\boldsymbol{r}}_z L^{\boldsymbol{p}}_\zeta) \cap L^2([0,\infty); L^{\frac{30}{11}}_z L^{\frac{10}{7}}_\zeta)$ for any $(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{p}) \in \Lambda$. Moreover there exists $f_\infty \in L^3_{z,\zeta}$ such that

$$\|f(t) - f_\infty \circ e^{-tH_p}\|_{L^3_{z,\zeta}} \to 0 \quad \text{as} \quad t \to \infty. \tag{0.6}$$

Remark

Chen-Holmer '23: quantum many body system $\to$ Boltzmann equation in the mean field limit for $p(z,\zeta) = |\zeta|^2$.