Topological crystals: Independence of spectral properties with respect to reference systems

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Outline

1. Topological crystals
2. Analysis on topological crystals
3. Independence of spectrum wrt reference systems

Notations in general graphs

- $V(X)$: a set of vertices $\{x_1, x_2, \dots\}$.
- $E(X)$: a set of unoriented edges $\{e_1, e_2, \dots\}$.
  $x_1 \sim x_2$ if a pair $(x_1, x_2) \in E(X)$.
- $A(X)$: a set of oriented edges $\{e_1, \overline{e_1}, \dots\}$.
- $o(e)$: the origin vertex of e,
  $t(e)$: the terminal vertex of e,
  where $o(\overline{e}) = t(e),\ t(\overline{e}) = o(e)$.
- Subset of $E(X)$ and $A(X)$ for each $x \in V(X)$:
  - $E(X)_x := \{e \in E(X) \mid x \text{ is an endpoint of } e\}$,
  - $A(X)_x := \{e \in A(X) \mid o(e) = x\}$.

Topological crystals

X and $\mathfrak{X}$: connected graphs, where $\mathfrak{X}$ is a finite graph.
A graph morphism
$\omega : X \to \mathfrak{X}, \ \big(\omega : V(X) \to V(\mathfrak{X}), \ \omega : E(X) \to E(\mathfrak{X})\big)$
s.t. if $x_1 \sim x_2$, then $\omega(x_1) \sim \omega(x_2)$.

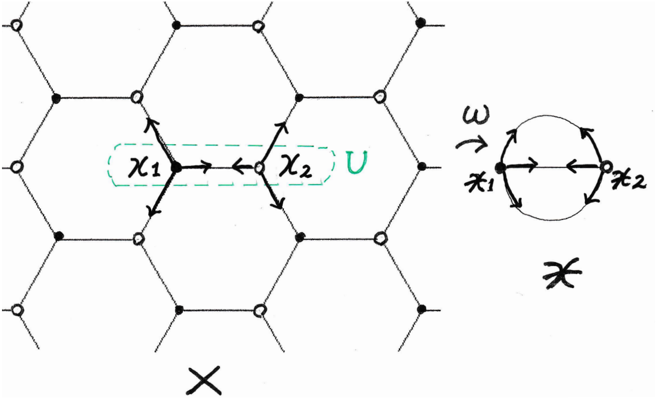
Definition 1 (Covering map, [4])

A morphism $\omega : X \to \mathfrak{X}$ is called a covering map,
X: a covering graph over the base graph $\mathfrak{X}$, if

1. $\omega : V(X) \to V(\mathfrak{X})$ *is surjective,*
2. $\forall x \in V(X), \ \omega|_{E(X)_x} : E(X)_x \to E(\mathfrak{X})_{\omega(x)}$ *is a bijection.*

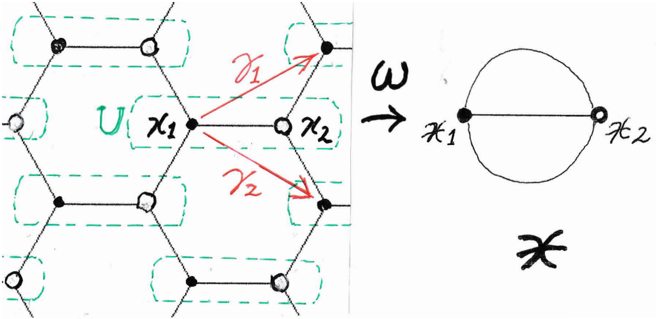
A Topological crystal and a Unit cell

Notations: $V(\mathfrak{X}) = \{\mathfrak{x}_1, \ldots, \mathfrak{x}_n\}$, $E(\mathfrak{X}) = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_k\}$.
We choose a unit cell $U = \{x_1, \ldots, x_n\} \subset V(X)$ satisfying $\omega(x_j) = \mathfrak{x}_j \ \forall j \in \{1, \ldots, n\}$.



The Transformation group

$\Gamma \subset \mathrm{Aut}(X)$ is called a *transformation group* acting upon X, s.t. $\omega \circ \gamma = \omega \ \forall \gamma \in \Gamma$.
We choose a set of generators $(\gamma_j) \equiv (\gamma_1, \dots, \gamma_d)$ of Γ.
d is the number of generators.



Definition 2 (Topological Crystals, [4])

A d-dimensional topological crystal is a quadruplet $(X, \mathfrak{X}, \omega, \Gamma)$ such that:

1. *X and $\mathfrak{X}$ are graphs, with $\mathfrak{X}$ finite,*
2. *$\omega : X \to \mathfrak{X}$ is a covering map,*
3. *The transformation group Γ of ω is isomorphic to $\mathbb{Z}^d$,*
4. *ω is regular, i.e. for every $x, y \in V(X)$ satisfying $\omega(x) = \omega(y)$, there exists $\gamma \in \Gamma$ such that $x = \gamma y$.*

Analysis on topological crystals; Periodic weight and degree function

Define a periodic weight
$m : V(X) \to (0, \infty),\ E(X) \to (0, \infty)$ s.t.
$m(x) = m(\gamma x),\ m(e) = m(\gamma e)$.
And set
$m(\mathfrak{x}) \equiv m(\omega(x)) := m(x),\ m(\mathfrak{e}) \equiv m(\omega(e)) := m(e)$,
and $m(\overline{e}) := m(e)$.
We introduce the degree function $\forall x \in V(X)$,

$$\deg_m(x) := \sum_{e \in A(X)_x} \frac{m(e)}{m(x)} = \sum_{\mathfrak{e} \in A(\mathfrak{X})_{\mathfrak{x}}} \frac{m(\mathfrak{e})}{m(\mathfrak{x})} =: \deg_m(\mathfrak{x})$$

The Laplace operator on topological crystals

Definition 3

The Laplace operator on a topological crystal $(X, \mathfrak{X}, \omega, \Gamma)$ *is defined on* $f \in \ell^2(X, m)$ *and for any* $x \in V(X)$ *by*

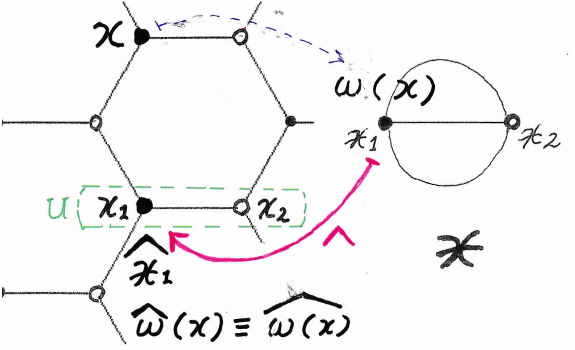
$$
\begin{aligned}
[\Delta(X, m)f](x) &:= \sum_{e \in A(X)_x} \frac{m(e)}{m(x)} \big(f(t(e)) - f(x)\big) \\
&= \sum_{e \in A(X)_x} \frac{m(e)}{m(x)} f(t(e)) - \mathrm{deg}_m(x) f(x).
\end{aligned}
$$

$\Delta(X, m)$ is a self-adjoint operator on the Hilbert space $\ell^2(X, m)$.

Three maps in topological crystals depending on U, (I) Lift

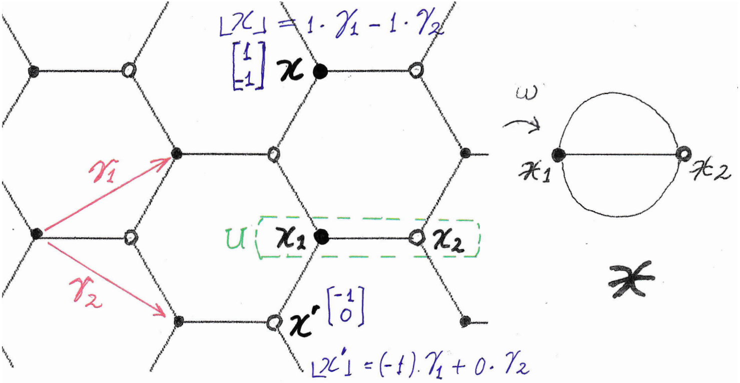
Lift $\widehat{\cdot} : V(\mathfrak{X}) \to U$ given by $\widehat{\mathfrak{x}_j} = x_j \in U$ for $\mathfrak{x}_j \in V(\mathfrak{X})$.

$\widehat{\omega}(\cdot) \equiv \widehat{\omega(\cdot)} : V(X) \to U$.



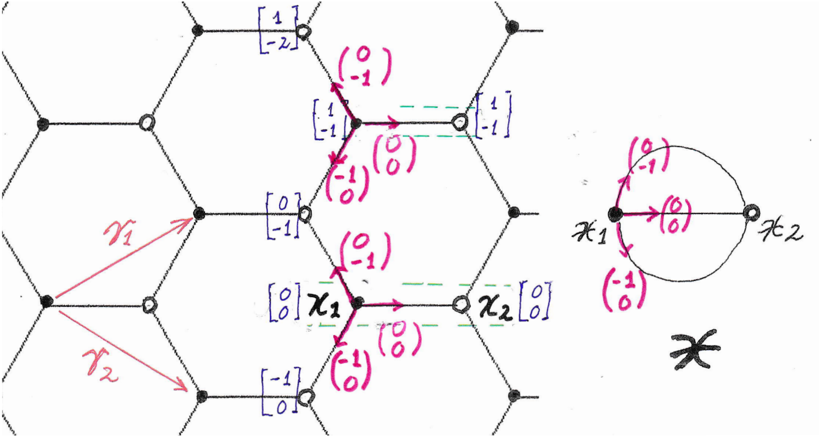
(2) The floor function

$\lfloor \cdot \rfloor : V(X) \to \Gamma$ given by $\lfloor x \rfloor \widehat{\omega}(x) = x \quad \forall x \in V(X)$.
We identify Γ with $\mathbb{Z}^d$ through generators (γ_j) as
$\lfloor x \rfloor = (\sum_{j=1}^{d} \mu_j \gamma_j)(x), \;\; \mu_j \in \mathbb{Z}$.



(3) The Index map

We define the index map $\eta : A(X) \to \Gamma$ given by
$\eta(e) := \lfloor t(e) \rfloor - \lfloor o(e) \rfloor \quad \forall e \in A(X).$



Three unitary transforms

1. $\mathcal{U}_U : \ell^2(X, m) \to \ell^2\left(\mathbb{Z}^d; \ell^2(\mathfrak{X})\right)$;
$$[\mathcal{U}_U f](\mu, \mathfrak{x}_j) = f(\mu x_j) \quad \mu \in \mathbb{Z}^d,$$
with μ, coordinate of $\lfloor x \rfloor$ wrt the generators of Γ.
2. $\mathcal{F} : \ell^2\left(\mathbb{Z}^d; \ell^2(\mathfrak{X})\right) \to L^2\left(\mathbb{T}^d; \ell^2(\mathfrak{X})\right)$;
$$[\mathcal{F}h](\zeta) = \sum_{\mu \in \mathbb{Z}^d} e^{-2\pi i \zeta \cdot \mu} h(\mu) \quad \zeta \in \mathbb{T}^d,$$
3. $\mathcal{I}_m : L^2\left(\mathbb{T}^d; \ell^2(\mathfrak{X})\right) \to L^2(\mathbb{T}^d; \mathbb{C}^n)$;
$$[\mathcal{I}_m q](\zeta) := \left(m(\mathfrak{x}_1)^{\frac{1}{2}} q(\zeta, \mathfrak{x}_1), \ldots, m(\mathfrak{x}_n)^{\frac{1}{2}} q(\zeta, \mathfrak{x}_n)\right)^T.$$

The product of unitary transforms is a unitary transform.

$$\mathcal{I}m\mathcal{F}\mathcal{U}_U : \ell^2(X, m) \to L^2(\mathbb{T}^d; \mathbb{C}^n). \tag{1}$$

For $g \in L^2(\mathbb{T}^d; \mathbb{C}^n)$,

$$[\mathcal{I}\mathcal{F}\mathcal{U}_U \Delta(X, m)[\mathcal{I}\mathcal{F}\mathcal{U}_U]^* g](\zeta) = h_U(\zeta) g(\zeta), \tag{2}$$

where $h_U(\zeta)$ is an $n \times n$ matrix with the entries given by

$$(h_U)_{jk}(\zeta) := \sum_{\mathfrak{e} \in A(\mathfrak{X}),\ o(\mathfrak{e}) = \mathfrak{x}_j,\ t(\mathfrak{e}) = \mathfrak{x}_k} \frac{m(\mathfrak{e})}{m(\mathfrak{x}_j)^{\frac{1}{2}} m(\mathfrak{x}_k)^{\frac{1}{2}}} e^{2\pi i \zeta \cdot \eta(\mathfrak{e})} - \deg(\mathfrak{x}_j) \delta_{jk}, \tag{3}$$

$\zeta \in \mathbb{T}^d$. The image of $\Delta(X, m)$ in $L^2(\mathbb{T}^d; \mathbb{C}^n)$ is a matrix-valued multiplication operator H_U defined by the continuous function $h_U : \mathbb{T}^d \to M_n(\mathbb{C})$.

Independence of spectrum wrt the Unit Cell

For distinct unit cells
$U = \{x_1, \ldots, x_n\}$, $U' = \{x'_1, \ldots, x'_n\}$ in X s.t.
$\omega(x_j) = \omega(x'_j) = \mathfrak{x}_j \quad j = 1, \ldots, n,$
distinct lifts and floor functions based on U and U';

$$\widehat{\cdot} : \mathfrak{X} \to U \quad \text{and} \quad \widehat{\cdot}\,' : \mathfrak{X} \to U',$$

$$\lfloor \cdot \rfloor : V(X) \to \Gamma \quad \text{s.t. } \lfloor x \rfloor \widehat{\omega}(x) = x,$$

and

$$\lfloor \cdot \rfloor' : V(X) \to \Gamma \quad \text{s.t. } \lfloor x \rfloor' \widehat{\omega}'(x) = x.$$

Then, distinct η and η' are defined accordingly.

Since $\mathcal{IFU}_U$ is a unitary transform,

$$\sigma\big([\mathcal{IFU}_U]\Delta(X,m)[\mathcal{IFU}_U]^*\big) = \sigma\big(\Delta(X,m)\big). \quad (4)$$

Only $\mathcal{U}_U$ depends on the choice of U. Let $\mathcal{U}_{U'}$ be a unitary operator for another unit cell U', then

$$\sigma\big([\mathcal{IFU}_U]\Delta(X,m)[\mathcal{IFU}_U]^*\big) = \sigma\big(\Delta(X,m)\big)$$

$$= \sigma\big([\mathcal{IFU}_{U'}]\Delta(X,m)[\mathcal{IFU}_{U'}]^*\big).$$

Thus,

$$\sigma(H_U) = \sigma(H_{U'}). \quad (5)$$

We have more than this . . . .

Theorem 4 ([1], Theorem 1.)

Let $h_U(\zeta)$ *and* $h_{U'}(\zeta)$ *denote the matrices computed wrt the unit cells* $U = \{x_1, \ldots, x_n\}$ *and* $U' = \{x_1', \ldots, x_n'\}$ *respectively. Then the equality*

$$\sigma\left(h_U(\zeta)\right) = \sigma\left(h_{U'}(\zeta)\right) \tag{6}$$

holds for any $\zeta \in \mathbb{T}^d$.

(6) is the equality for each ζ, which implies the equality for the union over all ζ. This is stronger than (5) which is the equality as a set.

Proof. By a small computation, one has

$$(h_{U'})_{jk}(\zeta) = e^{2\pi i \zeta\cdot\lfloor x'_j\rfloor}(h_U)_{jk}(\zeta)\ e^{-2\pi i \zeta\cdot\lfloor x'_k\rfloor}.$$

Define $V(\zeta) := \begin{pmatrix} e^{2\pi i\zeta\cdot\lfloor x'_1\rfloor} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i\zeta\cdot\lfloor x'_n\rfloor} \end{pmatrix}$.

Then, $h_{U'}(\zeta) = V(\zeta)h_U(\zeta)V(\zeta)^{-1}$.

Since $V(\zeta)$ is a unitary matrix,

$$\sigma\big(h_{U'}(\zeta)\big) = \sigma\big(V(\zeta)h_U(\zeta)V(\zeta)^{-1}\big) = \sigma\big(h_U(\zeta)\big).$$

(Q.E.D.)

Independence of spectrum wrt Generators for Γ

Let (γ_j) and (γ'_j) be distinct generators for Γ.
Then, there exist
$\alpha_{jk} \in \mathbb{Z}$ s.t. $\gamma_j = \sum_{k=1}^{d} \alpha_{jk} \gamma'_k \quad j = 1, \ldots, d$.
Set jk-matrix $\alpha = \left(\alpha_{jk}\right)$.
Let $\eta(e)$ and $\eta'(e)$ be the indices wrt (γ_j) and (γ'_j).
Then, by a computation,

$$\zeta \cdot \eta'(e) = (\alpha\zeta) \cdot \eta(e). \tag{7}$$

Let $h_U(\zeta)$ and $h'_U(\zeta)$ be matrices wrt (γ_j) and (γ'_j).

Theorem 5 ([1], Theorem 2.)

In the above framework,

$$\sigma\big(\Delta(X,m)\big) = \bigcup_{\zeta\in\mathbb{T}^d} \sigma\big(h_U(\zeta)\big) = \bigcup_{\zeta\in\mathbb{T}^d} \sigma\big(h'_U(\zeta)\big),$$

and for any $\zeta \in \mathbb{T}^d$, $\quad \sigma\big(h'_U(\zeta)\big) = \sigma\big(h_U(\alpha\zeta)\big)$,

but in general $\quad \sigma\big(h'_U(\zeta)\big) \neq \sigma\big(h_U(\zeta)\big)$.

That is, only the entire spectrum of $\Delta(X, m)$ is invariant under the change in the set of generators.

Idea of the proof

$$\sigma\left(h_U(\zeta)\right)$$

$$= \sigma\left(\left(\sum_{\mathfrak{e}\in A(\mathfrak{X}),\ o(\mathfrak{e})=\mathfrak{x}_j,\ t(\mathfrak{e})=\mathfrak{x}_k} \frac{m(\mathfrak{e})}{m(\mathfrak{x}_j)^{\frac{1}{2}} m(\mathfrak{x}_k)^{\frac{1}{2}}} e^{2\pi i \zeta\cdot\eta(\mathfrak{e})} - \deg(\mathfrak{x}_j)\delta_{jk}\right)_{jk}\right),$$

$$\sigma\left(h'_U(\zeta)\right)$$

$$= \sigma\left(\left(\sum_{\mathfrak{e}\in A(\mathfrak{X}),\ o(\mathfrak{e})=\mathfrak{x}_j,\ t(\mathfrak{e})=\mathfrak{x}_k} \frac{m(\mathfrak{e})}{m(\mathfrak{x}_j)^{\frac{1}{2}} m(\mathfrak{x}_k)^{\frac{1}{2}}} e^{2\pi i (\alpha\zeta)\cdot\eta(\mathfrak{e})} - \deg(\mathfrak{x}_j)\delta_{jk}\right)_{jk}\right).$$

Conclusion

We investigated the independence of the spectrum of the Laplace operator on topological crystal with distinct unit cells and distinct sets of generators of the transformation group.

1. The choice of the unit cell has only a weak impact on h_U, since the spectrum of h_U is preserved for each individual $\zeta \in \mathbb{T}^d$.
2. For the change in generators of the transformation group, only the entire spectrum of the Laplace operator is invariant.

References

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Thank you for your attention!