Uniform resolvent and smoothing estimates for the Heisenberg sub-Laplacian

Haruya Mizutani (Osaka University)

Joint work with
Luca Fanelli (Bilbao), Luz Roncal (Bilbao),
and Nico Michele Schiavone (Madrid)

Consider first the free Schrödinger resolvent and group on $\mathbb{R}^d$:

$$(-\Delta_{\mathbb{R}^d} - \sigma)^{-1}, \quad e^{it\Delta_{\mathbb{R}^d}}$$

- Uniform resolvent estimate with a closed operator G

$$\text{(Resolvent)} \qquad \sup_{\sigma \in \mathbb{C}\setminus\mathbb{R}} \|G(-\Delta_{\mathbb{R}^d} - \sigma)^{-1} G^*\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} < \infty$$

- Local smoothing effects with G:

$$\text{(Smoothing)} \quad \begin{cases} \|Ge^{it\Delta_{\mathbb{R}^d}} f\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}, \\ \left\| \int_0^t Ge^{i(t-s)\Delta_{\mathbb{R}^d}} G^* F(s) ds \right\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \end{cases}$$

- It is known (Resolvent) $\Leftrightarrow$ (Smoothing) (Kato '66, Mochizuki '11)

Some examples of (R) and (S) for $\Delta_{\mathbb{R}^d}$

$$\text{(Resolvent)} \quad \sup_{\sigma\in\mathbb{C}\setminus\mathbb{R}} \|G(-\Delta_{\mathbb{R}^d}-\sigma)^{-1}G^*\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} < \infty$$

$$\text{(Smoothing)} \begin{cases} \|Ge^{it\Delta_{\mathbb{R}^d}}f\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}, \\ \left\|\int_0^t Ge^{i(t-s)\Delta_{\mathbb{R}^d}}G^*F(s)ds\right\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \end{cases}$$

- (Resolvent) hold for
  - $G = |x|^{\alpha-1}|D|^{\alpha}$ $(1-\frac{d}{2} < \alpha < \frac{1}{2})$.
    (Kato–Yajima '89, Watanabe '91, Sugimoto '98)
  - $G = \langle x\rangle^{-\mu}|D|^{\frac{1}{2}}$ $(\mu > \frac{1}{2}, d \geq 2)$.

  (Kenig–Ponce–Vega '91, Ben-Artzi–Klainerman '92, Chihara '02)
- Application: Ex) If $d \geq 3$, $H = -\Delta_{\mathbb{R}^d} + \mu V(x)$, V is $\mathbb{R}$-valued, $|V| \lesssim |x|^{-2}$, $|\mu| \ll 1$, (Resolvent) holds with $G = |V(x)|^{1/2}$, then $W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{it\Delta_{\mathbb{R}^d}}$ exist and $\mathrm{Ran}\, W_\pm = L^2(\mathbb{R}^d)$

Two generalizations

- Elliptic case: if $a \in C^\infty(\mathbb{R}^d \setminus 0)$, $a(\lambda\xi) = \lambda^m a(\xi)$ $(\lambda, m > 0)$ and $\nabla a(\xi) \neq 0$ for $\xi \neq 0$, then for $\mu > \frac{1}{2}$

  $$\sup_{\sigma \in \mathbb{C} \setminus \mathbb{R}} \| \langle x \rangle^{-\mu} |D|^{\frac{m-1}{2}} (a(D) - \sigma)^{-1} |D|^{\frac{m-1}{2}} \langle x \rangle^{-\mu} \| < \infty.$$

  (Chihara '02, Ruzhansky–Sugimoto '12)
- Non-elliptic case: $P = -\partial_{x_1}^2 - x_1^2 \partial_{x_2}^2$ (Grushin operator) on $\mathbb{R}^2$,

  $$\sup_{\sigma \in \mathbb{C} \setminus \mathbb{R}} \| \langle A \rangle^{-\mu} P^{1/2} (P - \sigma)^{-1} P^{1/2} \langle A \rangle^{-\mu} \| < \infty, \mu > \frac{1}{2}$$

  where $A = \frac{1}{2i}(x_1 \partial_{x_1} + \partial_{x_1} x_1 + 2x_2 \partial_{x_2} + 2\partial_{x_2} x_2)$. (Hoshiro '99)

Today's topic

Generalization of Hoshiro's result to the Heisenberg sub-Laplacian

2. Heisenberg group and sub-Laplacian

- Heisenberg group: $\mathbb{H}^d = \mathbb{R}^{2d+1}$ with the group law
  $$(z,s)\cdot(z',s') = (x+x', y+y', s+s'+2x'\cdot y - 2x\cdot y')$$
  where $z=(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ and $s\in\mathbb{R}$.
- Left invariant vector fields:
  $$X_j = \partial_{x_j} + 2y_j\partial_s, \quad Y_j = \partial_{y_j} - 2x_j\partial_s, \quad \partial_s = [Y_1, X_1]/4$$
- Sub-Laplacian: $\nabla_{\mathbb{H}} = (X_1, ..., X_d, Y_1, ..., Y_d)$
  $$\Delta_{\mathbb{H}} = \nabla_{\mathbb{H}}\cdot\nabla_{\mathbb{H}} = \sum_{j=1}^{d}(X_j^2 + Y_j^2) = \Delta_z + 4N\partial_s + 4|z|^2\partial_s^2,$$
  where $N := z^\perp\cdot\nabla_z$, $z^\perp = (y,-x)$ for $z=(x,y)$.
- $\Delta_{\mathbb{H}}$ is self-adjoint on $L^2(\mathbb{H}^d) = L^2(\mathbb{R}^{2d+1}, dzds)$.
- $\sigma(\Delta_{\mathbb{H}}) = \sigma_{\mathrm{ac}}(\Delta_{\mathbb{H}}) = (-\infty, 0], \sigma_{\mathrm{p}}(\Delta_{\mathbb{H}}) = \sigma_{\mathrm{sc}}(\Delta_{\mathbb{H}}) = \emptyset.$

Motivation: Non-dispersiveness of the Schrödinger flow

Bahouri–Gérard–Xu '00: Let

$$f_*(z,s) := \int_{\mathbb{R}} e^{is\lambda} e^{-\lambda|z|^2} g(\lambda) d\lambda, \quad g \in C_0^\infty((0,\infty))$$

Then $e^{it\Delta_{\mathbb{H}}} f_*$ is given by the translation in s-direction:

$$e^{it\Delta_{\mathbb{H}}} f_*(z,s) = f_*(z, s-4td)$$

- $\|e^{it\Delta_{\mathbb{H}}} f_*\|_{L^p(\mathbb{H}^d)} = \|f_*\|_{L^p(\mathbb{H}^d)}$ for any p and t.
- No global dispersive (time-decay) nor Strichartz estimates
- No local smoothing effects with respect to ∂_z or ∂_s.

However, with $L_s^\infty L_t^2 L_z^2 = L^\infty(\mathbb{R}_s; L^2(\mathbb{R}_t \times \mathbb{R}_z^{2d}))$,

$$\|e^{it\Delta_{\mathbb{H}}} f_*\|_{L_s^\infty L_t^2 L_z^2} = (2\sqrt{d})^{-1} \|f_*\|_{L^2(\mathbb{H}^d)},$$

$$\||z|\nabla_{\mathbb{H}} e^{it\Delta_{\mathbb{H}}} f_*\|_{L_s^\infty L_t^2 L_z^2} = C_d \|f_*\|_{L^2(\mathbb{H}^d)}$$

with $C_d = 2^{-1/2}(d+1)^{1/2}\pi^{-d/2}\Gamma(d/2)$.

Known results

- Măntoiu '17: $L^2_{t,z,s}$-estimates with derivative loss

  $$\|\langle |(z,s)|_{\mathbb{H}}\rangle^{-2\mu}\langle\nabla_{\mathbb{H}}\rangle^{-\mu}|\nabla_{\mathbb{H}}|^{1-\mu}e^{it\Delta_{\mathbb{H}}}f\|_{L^2(\mathbb{R}\times\mathbb{H}^d)}\lesssim\|f\|_{L^2(\mathbb{H}^d)}.$$

  where $1/2<\mu\leq 1$ and $|(z,t)|_{\mathbb{H}}=(|z|^4+s^2)^{1/4}$.
- Bahouri–Barilari–Gallagher '21:

  Strichartz-type estimates for f radial in z:

  $$\|e^{it\Delta_{\mathbb{H}}}f\|_{L^\infty_s L^p_t L^q_z}\lesssim\|f\|_{L^2},\quad f(z,s)=f(|z|,s)$$

  for $2\leq p,q\leq\infty$, $2/p+2d/q=d+1$. In particular

  $$\|e^{it\Delta_{\mathbb{H}}}f\|_{L^\infty_s L^2_t L^2_z}+\|\langle s\rangle^{-1/2-}e^{it\Delta_{\mathbb{H}}}f\|_{L^2_{t,z,s}}\lesssim\|f\|_{L^2}.$$
- Barilari–Flynn (preprint '25): general f, non-endpoint $p>2$
- Bahouri–Gallagher '23: Explicit kernel formula for $e^{it\Delta_{\mathbb{H}}}$, local-in-time dispersive and Strichartz esti.

3. Result (Fanelli–M. –Roncal–Schiavone, preprint '24)

Theorem 1 (Resolvent estimates)

Consider the following weights G_j:

- $d \geq 1$, $G_1 = \langle z \rangle^{-1} \langle |(z,s)|_{\mathbb{H}} \rangle^{-2} |z|$
- $d \geq 2$, $G_2 = (1 + |z|^2 + s^2)^{-1/2}$
- $d \geq 2$, $\mu > \frac{1}{2}$, $G_3 = \langle N \rangle^{-\mu} \langle |(z,s)|_{\mathbb{H}} \rangle^{-2\mu} |z|^{\mu} |\nabla_{\mathbb{H}}|^{\frac{1}{2}}$
- $d \geq 2$, $\frac{1}{2} < \mu \leq 1$, $G_4 = \langle N \rangle^{-\mu} |(z,s)|_{\mathbb{H}}^{-2\mu} |z|^{\mu} |\nabla_{\mathbb{H}}|^{1-\mu}$,

where $N = z^{\perp} \cdot \nabla_z$, $z^{\perp} = (y, -x)$ *for* $z = (x, y)$. *Then*

$$\sup_{\sigma \in \mathbb{C} \setminus \mathbb{R}} \| G_j (-\Delta_{\mathbb{H}} - \sigma)^{-1} G_j^* f \|_{L^2(\mathbb{H}^d)} \lesssim \| f \|_{L^2(\mathbb{H}^d)}$$

for $j = 1, \ldots, 4$. *Moreover, if* $f(z,s)$ *is radial with respect to* z, *then* $\langle N \rangle^{-\mu}$ *in* G_3 *and* G_4 *can be removed.*

Result (Fanelli–M. –Roncal–Schiavone, preprint '24)

- $d \geq 1,\ G_1 = \langle z\rangle^{-1}\langle |(z,s)|_{\mathbb{H}}\rangle^{-2}|z|$
- $d \geq 2,\ G_2 = (1 + |z|^2 + s^2)^{-1/2}$
- $d \geq 2,\ \mu > \frac{1}{2},\ G_3 = \langle N\rangle^{-\mu}\langle |(z,s)|_{\mathbb{H}}\rangle^{-2\mu}|z|^{\mu}|\nabla_{\mathbb{H}}|^{\frac{1}{2}}$
- $d \geq 2,\ \frac{1}{2} < \mu \leq 1, G_4 = \langle N\rangle^{-\mu}|(z,s)|_{\mathbb{H}}^{-2\mu}|z|^{\mu}|\nabla_{\mathbb{H}}|^{1-\mu}$

Theorem 2 (Kato smoothing effects)

Under the same condition in Theorem 1, for $j = 1, ..., 4$,

$$\|G_j e^{it\Delta_{\mathbb{H}}} f\|_{L^2(\mathbb{R}\times\mathbb{H}^d)} \lesssim \|f\|_{L^2(\mathbb{H}^d)}$$

$$\left\| G_j \int_0^t e^{i(t-r)\Delta_{\mathbb{H}}} G_j^* F(r) dr \right\|_{L^2(\mathbb{R}\times\mathbb{H}^d)} \lesssim \|F\|_{L^2(\mathbb{R}\times\mathbb{H}^d)}.$$

- $\langle N\rangle^{-\mu}$ is a kind of spherical derivative losses ($N = z^{\perp} \cdot \nabla_z$)
- If f and F are radial in z, we have similar $\frac{1}{2}$-local smoothing effects to the Euclidean case.

4. Idea of the proof. 1: The method of weakly conjugate operators (Boutet de Monvel–Măntoiu '97, Hoshiro '99)

Let $A = \frac{1}{2i}(z \cdot \nabla_z + \nabla \cdot z + 2s\partial_s + 2\partial_s s)$ be the generator of

$$e^{i\lambda A} f(z,s) = e^{(d+1)\lambda} f(e^{\lambda} z, e^{2\lambda} s), \quad \lambda \in \mathbb{R}$$

such that $[\Delta_{\mathbb{H}}, iA] = 2\Delta_{\mathbb{H}}$. Let $\alpha \in \mathbb{R}$ and

$$A_\alpha = \langle \nabla_{\mathbb{H}} \rangle^{-\alpha} A \langle \nabla_{\mathbb{H}} \rangle^{-\alpha}, \quad W_\alpha = \langle A_\alpha \rangle^{-\mu} |\nabla_{\mathbb{H}}| \langle \nabla_{\mathbb{H}} \rangle^{-\alpha}$$

We apply essentially the same differential inequality method as in Mourre '81 to the operator

$$F(\varepsilon) = \langle \varepsilon A_\alpha \rangle^{\mu-1} W_\alpha (-\Delta_{\mathbb{H}} - \sigma + i\varepsilon[\Delta_{\mathbb{H}}, iA])^{-1} W_\alpha^* \langle \varepsilon A_\alpha \rangle^{\mu-1}$$

with $\varepsilon > 0$ and $\operatorname{Im} \sigma > 0$ to obtain

$$\sup_{\sigma \in \mathbb{C} \setminus \mathbb{R}} \| W_\alpha (-\Delta_{\mathbb{H}} - \sigma)^{-1} W_\alpha^* \| < \infty, \quad \mu > 1/2, \ \alpha \geq 0.$$

We have obtained $\sup_{\sigma\in\mathbb{C}\setminus\mathbb{R}}\|W_\alpha(-\Delta_{\mathbb{H}}-\sigma)^{-1}W_\alpha^*\|<\infty$

It remains to replace W_α by G_j with appropriate α depending on j. Consider the case $G_2=(1+|z|^2+s^2)^{-1/2}$. Then

$$G_2 = G_2|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle(iA_1+1)\cdot\underbrace{(iA_1+1)^{-1}\langle A_1\rangle}_{\text{bounded}}\cdot\underbrace{\langle A_1\rangle^{-1}|\nabla_{\mathbb{H}}|\langle\nabla_{\mathbb{H}}\rangle^{-1}}_{=W_1 \text{ with } \mu=1},$$

$$\begin{aligned}G_2|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle(iA_1+1) &= G_2|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle(i\langle\nabla_{\mathbb{H}}\rangle^{-1}A\langle\nabla_{\mathbb{H}}\rangle^{-1}+1)\\ &= \underbrace{G_2 z\cdot\nabla_{\mathbb{H}}|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle^{-1}}_{\text{bounded}}+2\underbrace{G_2 s}_{\text{bounded}}\partial_s|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle^{-1}\\ &\quad + dG_2|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle^{-1}+G_2|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle\end{aligned}$$

Thus, it is enough to observe

$$\partial_s|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle^{-1},\ G_2|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle^{-1},\ G_2|\nabla_{\mathbb{H}}|^{-1}\langle\nabla_{\mathbb{H}}\rangle\in\mathbb{B}(L^2)$$

4. Idea of the proof. 2: Hardy's type inequalities

- $\partial_s |\nabla_{\mathbb{H}}|^{-1} \langle \nabla_{\mathbb{H}} \rangle^{-1} \in \mathbb{B}(L^2)$: We use the sub-elliptic estimate

$$\|\partial_s f\| \leq \frac{1}{2} \|\Delta_{\mathbb{H}} f\|,$$

which follows from the facts $\partial_s = \frac{1}{4}[X_1, Y_1]$ and $[\partial_s, \Delta_{\mathbb{H}}] = 0$:

$$\|\partial_s (-\Delta_{\mathbb{H}})^{-1}\| \leq \frac{1}{4} \| |\nabla_{\mathbb{H}}|^{-1} [X_1, Y_1] |\nabla_{\mathbb{H}}|^{-1} \| \leq \frac{1}{2}$$

- $G_2 |\nabla_{\mathbb{H}}|^{-1} \langle \nabla_{\mathbb{H}} \rangle^{-1}, G_2 |\nabla_{\mathbb{H}}|^{-1} \langle \nabla_{\mathbb{H}} \rangle \in \mathbb{B}(L^2)$: By Hardy's type inequality (D'Ambrosio '04)

$$\| |z|^{-1} f \| \leq (d-1)^{-1} \|\nabla_{\mathbb{H}} f\|,$$

we obtain $\|G_2 |\nabla_{\mathbb{H}}|^{-1}\| \lesssim 1$ and

$$\|G_2 |\nabla_{\mathbb{H}}|^{-1} \langle \nabla_{\mathbb{H}} \rangle\| \leq \|G_2 |\nabla_{\mathbb{H}}|^{-1} \langle \nabla_{\mathbb{H}} \rangle \chi_{[0,1]}(-\Delta_{\mathbb{H}})\| + \|G_2 |\nabla_{\mathbb{H}}|^{-1} \langle \nabla_{\mathbb{H}} \rangle \chi_{[1,\infty]}(-\Delta_{\mathbb{H}})\| \lesssim 1$$

Future topics

1. On resolvent and smoothing estimates on $\mathbb{H}^d$:

- Potential perturbation $H = -\Delta_{\mathbb{H}} + V(z,s)$
- Generalization to more general stratified Lie group
- Endpoint Strichartz-type estimates with f not radial in z:

$$\|e^{it\Delta_{\mathbb{H}}} f\|_{L^\infty_s L^2_{t,z}} \lesssim \|f\|_{L^2}?$$

2. On spectral and scattering theory on $\mathbb{H}^d$:

- Radiation condition: We can prove *a priori* estimate:

$$\|\nabla_{\mathbb{H}}(e^{-i\psi}u)\|_{L^2} \lesssim \| |(z,s)|^2_{\mathbb{H}} |z|^{-1} f\|_{L^2}, \quad 0 < \varepsilon \lesssim \lambda$$

if $u = (-\Delta_{\mathbb{H}} - \lambda - i\varepsilon)^{-1} f$ satisfies $u(z,s) = u(|(z,s)|_{\mathbb{H}})$, where $\psi(z,s) = \sqrt{\frac{d}{2}\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}}\sqrt{\lambda}|(z,s)|_{\mathbb{H}}$

- Structures of geometric scattering matrices

Thank you for your attention