Rellich's theorem for Dirac equation with long-range perturbations

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Aim

In this talk, we prove Rellich's theorem for the Dirac equation.

This is a joint work with Prof. Ito Kenichi, who gave the previous talk.

Definition (Dirac equation)

First, we define the Dirac equation on the Euclidean space $\mathbb{R}^d$ of $d \in \mathbb{N} \setminus \{0\}$. The Dirac equation is given by:

$$\partial_t \psi = -iH\psi$$

where

$$H = \alpha_j p_j + q, \; p_j = -i\partial_j$$

for $j = 1, 2, \ldots, d$. Here, for some $n \in \mathbb{N} \setminus \{0\}$, ψ is an unknown $\mathbb{C}^n$-valued function and q is a given $n \times n$ Hermitian matrix-valued function: we refer to q as **the potential**.

(The Einstein summation convention is used throughout this slides.)

Definition (α_j)

$\alpha_1, \ldots, \alpha_n$ are $n \times n$ Hermitian matrices satisfying

$$\forall j, k = 1, \ldots, n,\ \{\alpha_j, \alpha_k\} = 2\delta_{jk} I$$

where I is the identity matrix.

It is well-known that such α_j exist if $n = 2^{\lfloor d/2 \rfloor}$, or if $n = 2^{\lfloor (d+1)/2 \rfloor}$ when β satisfying the following is added:

$$\forall j = 1, \ldots, n,\ \{\alpha_j, \beta\} = 0,\ \beta^2 = I.$$

However, our argument is independent of the representation of α_j or the choice of n; we rely solely on the anticommutation relations.

Assumption on the potential q

In order to prove Rellich's theorem, we assume some conditions on the potential q.

For this purpose, we introduce **a modified radius**.

Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 & (t \leq 1) \\ 0 & (t \geq 2) \end{cases}, \ \chi' \leq 0,$$

and define a modified radius f as

$$f = \chi(|x|) + |x|(1 - \chi(|x|)).$$

In addition, we define

$$\partial_f := (\partial_j f)\partial_j.$$

Assumption on the potential q

We denote the set of all $n \times n$ Hermitian matrices by $\mathbb{C}^{n\times n}_{\mathrm{Her}}$, and its norm, arbitrarily fixed one, by $|\cdot|$.

Now, we assume that the potential q splits as

$$q = q_0 + q_1 + q_2,\ q_0, q_1 \in (C^1 \cap L^\infty)(\mathbb{R}^d; \mathbb{C}^{n\times n}_{\mathrm{Her}}),\ q_2 \in L^2(\mathbb{R}^d; \mathbb{C}^{n\times n}_{\mathrm{Her}}),$$

such that the identities

$$\alpha_1 q_0 \alpha_1 = \alpha_2 q_0 \alpha_2 = \cdots = \alpha_d q_0 \alpha_d =: \tilde{q}_0$$

hold on $\mathbb{R}^d$.

Assumption on the potential q

Furthermore, we assume

$$\exists \rho \in \,]0,1], \exists C > 0,$$
$$|q_1| \leq Cf^{-(1+\rho)/2},$$
$$|\partial_1 q_0| + \cdots + |\partial_d q_0| + |\partial_f q_1| + |q_2| \leq Cf^{-(1+\rho)}$$

hold on $\mathbb{R}^d$.

Physical interpretation and an example of q_0

Physically, q_0 can be regarded as the sum of the mass and an electric potential that may not vanish at spatial infinity. In fact, if the electro-magnetic potential A is $(V, -A_1, \ldots, -A_d)$ and mass is m, then the Dirac equation can be written as

$$\partial_t \psi = -i(\alpha_j p_j + \beta m + V - \alpha_j A_j)\psi.$$

In this case, we get

$$q_0 = \beta m + V - \alpha_j A_j.$$

It is a straightforward calculation to verify that this satisfies the aforementioned conditions.

Definition (Function space)

Next, we are going to define **the Agmon-Hörmander spaces** $\mathcal{B}, \mathcal{B}^*, \mathcal{B}_0^*$.

Let $F(S)$ be the indicator for a given subset $S \subset \mathbb{R}^d$, and set

$$F_\nu := F(\{x \in \mathbb{R}^d \mid 2^\nu \leq f(x) < 2^{\nu+1}\}).$$

Using this, we define $\mathcal{B}, \mathcal{B}^*, \mathcal{B}_0^*$ as follows:

$$\mathcal{B} = \{\psi \in L^2(\mathbb{R}^d; \mathbb{C}^n) \mid \|\psi\|_{\mathcal{B}} < \infty\}, \ \|\psi\|_{\mathcal{B}} := \sum_{\nu \in \mathbb{N}} 2^{\nu/2} \|F_\nu \psi\|_{L^2},$$

$$\mathcal{B}^* = \{\psi \in L^2(\mathbb{R}^d; \mathbb{C}^n) \mid \|\psi\|_{\mathcal{B}^*} < \infty\}, \ \|\psi\|_{\mathcal{B}^*} := \sup_{\nu \in \mathbb{N}} 2^{-\nu/2} \|F_\nu \psi\|_{L^2},$$

$$\mathcal{B}_0^* = \{\psi \in \mathcal{B}^* \mid \lim_{\nu \to \infty} 2^{-\nu/2} \|F_\nu \psi\|_{L^2} = 0\}.$$

Main theorem

We set

$$m_+ := \liminf_{R\to\infty}\{\lambda \in \mathbb{R} \mid \forall |x| > R, \lambda \geq q_0(x)\},$$
$$m_- := \limsup_{R\to\infty}\{\lambda \in \mathbb{R} \mid \forall |x| > R, \lambda \leq q_0(x)\}$$

where the inequality involving λ above is understood in the sense of matrices, or as an inequality between quadratic forms on $\mathbb{C}^n$.

Main theorem

The main theorem is as follows:

Rellich's theorem

We impose the aforementioned assumptions on q. If $\phi \in \mathcal{B}_0^*$ and $\lambda \in \mathbb{R} \setminus [m_-, m_+]$ satisfy

$$(H - \lambda)\phi = 0$$

in the distributional sense, then $\phi = 0$ as a function on $\mathbb{R}^d$.

Previous research

Rellich's theorem for the Schrödinger equation is well-known for various potentials:

[1]:*Stationary scattering theory on manifolds,* K. Ito, E. Skibsted: long-range potential

[2]:*A Rellich type theorem for the generalized oscillator,* T. Tagawa: extra potential with the form $a|x|^b$, oscillator

[3]:*Rellich's theorem for spherically symmetric repulsive Hamiltonians,* K. Itakura: extra potential with the form $-|x|^{\varepsilon}$, repulsive

Strategy of the proof

According to *Spectral theory of Dirac operators with potential diverging at infinity* written by H. T. Ito and O. Yamada, if q is smooth and purely electrical, the problem can be reduced to the Schrödinger case. However, we do not employ that method here, and use the commutator argument of Ito-Skibsted [1].

Strategy of the proof of main theorem

First, we decompose the theorem into two parts. From now on, we assume that q enjoys the aforementioned assumptions.

Proposition 1

Let $\phi \in \mathcal{B}_0^*$ and $\lambda \in \mathbb{R} \setminus [m_-, m_+]$ satisfy $(H - \lambda)\phi = 0$ in the distributional sense, then $\forall \kappa > 0, e^{\kappa f}\phi \in \mathcal{B}_0^*$

Proposition 2

Let $\phi \in \mathcal{B}_0^*$ and $\lambda \in \mathbb{R} \setminus [m_-, m_+]$ satisfy $(H - \lambda)\phi = 0$ and $\forall \kappa > 0, e^{\kappa f}\phi \in \mathcal{B}_0^*$, then $\phi = 0$ as a function on $\mathbb{R}^d$

To prove the main theorem, we define the conjugate operator.
From now on, the constant $\delta \in]0, \rho[$ is arbitrarily fixed.

$$\bar{\chi}_a := 1 - \chi(f/2^a), \ \chi_b := \chi(f/2^b), \ \chi_{a,b} := \bar{\chi}_a \chi_b,$$

$$\theta = \theta_\nu^{\kappa,\mu} := 2\kappa f + \underbrace{2\mu \int_0^f (1 + \tau/2^\nu)^{-1-\delta} d\tau}_{\text{refinement of Yoshida approx.}},$$

$$\Theta := \chi_{a,b} e^\theta,$$

$$A_\Theta := 2\Re(\Theta p_f) = \Theta p_f + p_f^* \Theta,$$

where p_f^* is the adjoint of $p_f := (\partial_j f) p_j$.

Note that $\int_0^f (1 + \tau/2^\nu)^{-1-\delta} d\tau \uparrow f$ pointwise as $\nu \to \infty$.

Sketch of the proof of proposition 1

We present the key inequality for the proof of Proposition 1.

Lemma 1

Let $\lambda \gtrless m_{\pm}$, respectively, and fix any $\kappa_0 > 0$. Then there exist $\mu, c, C > 0$ and $a_0 \in \mathbb{N}$ such that uniformly in $\kappa \in [0, \kappa]$, $b > a \geq a_0$ and $\nu \in \mathbb{N}$

$$\begin{aligned}\pm\Im(A_\Theta(H-\lambda)) &\geq c\min\{f^{-1}, \theta'\}\Theta \\ &\quad - C(\chi^2_{a-1,a+1} + \chi^2_{b-1,b+1})f^{-1}e^{\theta} \\ &\quad - (H-\lambda)\gamma(H-\lambda)\end{aligned}$$

respectively, where $\gamma = \gamma_{a,b}$ is a certain function.

Sketch of the proof of proposition 1

Assume $\kappa_0 := \sup\{\kappa \geq 0 \mid e^{\kappa f}\phi \in \mathcal{B}_0^*\} < \infty$, and let $\mu > 0$ as in the Lemma 1. We set $\kappa = 0$ if $\kappa_0 = 0$, and $\kappa \in [0, \kappa_0[$ with $\kappa + \mu > \kappa_0$ otherwise.

Taking the expectation value of both sides of the inequality with respect to the eigenstate ϕ,

$$
\begin{aligned}
\langle \phi, \Im(A_\Theta \underbrace{(H-\lambda))\phi}_{=0}\rangle \geq{}& \langle \phi, c\min\{f^{-1}, \theta'\}\Theta\phi\rangle \\
&- \langle \phi, C(\chi^2_{a-1,a+1} + \underbrace{\chi^2_{b-1,b+1})f^{-1}e^{\theta}}_{\text{vanishes as } b\to\infty}\ \phi\rangle \\
&- \langle \phi, (H-\lambda)\gamma\underbrace{(H-\lambda)\phi}_{=0}\rangle
\end{aligned}
$$

Furthermore, by taking the limit $\nu \to \infty$, we replace θ by f, and we get $e^{(\kappa+\mu)f}\phi \in \mathcal{B}_0^*$. This is a contradiction.

Sketch of the proof of proposition 2

We present the key inequality for the proof of Proposition 2.

Lemma 2

Let $\lambda \gtrless m_\pm$, respectively, and fix $\mu = 0$ in the definition of Θ, so that $\theta = 2\kappa f$. Then there exist $c, C > 0$ and $a_0 \in \mathbb{N}$ such that uniformly in $\kappa \geq 1$ and $b > a \geq a_0$

$$\begin{aligned}\pm\Im(A_\Theta(H-\lambda)) &\geq c\kappa^2 \min\{f^{-1}, \theta'\}\Theta \\ &\quad - C\kappa^2(\chi^2_{a-1,a+1} + \chi^2_{b-1,b+1})f^{-1}e^{2\kappa f} \\ &\quad - (H-\lambda)\gamma(H-\lambda)\end{aligned}$$

respectively, where $\gamma = \gamma_{a,b}$ is a certain function.

Sketch of the proof of proposition 2

The details are omitted here, but the proof of proposition 2 is similar to that of proposition 1. It can be shown by taking the expectation value of both sides of lemma 2 with respect to $\chi_{a-2,b+2}\phi$.