Snapshot problem for the Klein-Gordon equation on spheres

joint with
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When does the above equation have a unique solution?

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We denote by $\{Y_{lm}\}$, $(l \in \mathbb{Z}_{\geq 0}, 1 \leq m \leq d(l))$ the set of spherical harmonics on S^n. (Note that $\{Y_{lm}\}$ forms a CONS in $L^2(S^n)$.)

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We do not need an explicit expression of Y_{lm}. All we need is

$$\Delta_{S^n} Y_{lm} = -l(l+n-1)Y_{lm}, \qquad (l \in \mathbb{Z}_{\geq 0}, 1 \leq m \leq d(l)).$$

Solution of the Cauchy problem for KG equation

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$$u(x,t) = \sum_{l \geq 0} \sum_{m=1}^{d(l)} \left(a_{lm} \cos \lambda_M(l) t + d_{lm} \frac{\sin \lambda_M(l) t}{\lambda_M(l)} \right) Y_{lm}(x).$$

where

$$\lambda_M(l) = \sqrt{\left(l + \frac{n-1}{2}\right)^2 + M^2}.$$

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$$u(x,t) = S_t' f_0(x) + S_t g_0(x).$$

where $S_t' = \partial_t S_t$.

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Uniqueness of the solution to the snapshot problem

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Summarizing the argument above, we have

Theorem

The uniqueness of the solution holds for the snapshot problem for the Klein-Gordon equation on S^n if and only if

$$(\#) \quad |\sin\lambda_M(l)t_1| + |\sin\lambda_M(l)t_2| > 0, \quad \textit{for}\ \forall\ l,$$

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(b) *We write* $t_1/t_2 = p/q$ *in lowest terms and put* $\tau := t_1/p = t_2/q$.
If n *is odd, then* $\left(\frac{\pi}{\tau}\right)^2 - M^2$ *is a square integer.*
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The proof is straightforward, so we omit it.

Compatibility condition

We note that the snapshot problem

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In fact, as we mentioned, for two given smooth functions f_0 and g_0, the Cauchy problem for the Klein-Gordon equation on S^n

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has a unique solution.

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$$(SP)\ \begin{cases} \left(\Delta_{S^n} - \left(\frac{n-1}{2}\right)^2 - M^2\right) u(x,t) = \frac{\partial^2}{\partial t^2}\, u(x,t), \\ u\big|_{t=0} = f_0,\ \ u\big|_{t=t_1} = f_1,\ \ u\big|_{t=t_2} = f_2. \end{cases}$$

is overdetermined.

In fact, as we mentioned, for two given smooth functions f_0 and g_0, the Cauchy problem for the Klein-Gordon equation on S^n

$$(CP)\ \begin{cases} \left(\Delta_{S^n} - \left(\frac{n-1}{2}\right)^2 - M^2\right) u(x,t) = \frac{\partial^2}{\partial t^2}\, u(x,t), \\ u(x,0) = f_0(x),\ \ \partial_t u(x,0) = g_0(x), \end{cases}$$

has a unique solution.

So we need a compatibility condition on three functions $f_0,\ f_1$ and f_2.

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we have

$$S_{t_2}(f_1 - S_{t_1}' f_0) = S_{t_2} S_{t_1} g_0 = S_{t_1} S_{t_2} g_0 = S_{t_1}(f_2 - S_{t_2}' f_0).$$

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Proposition (Compatibility condition on the snapshot data)

If the snapshot problem (SP) *with the snapshot data* $\{f_0,\ f_1,\ f_2\}$ *has a solution, then* $f_0,\ f_1$ *and* f_2 *satisfy the following compatibility condition.*

$$S_{t_2}(f_1 - S_{t_1}' f_0) = S_{t_1}(f_2 - S_{t_2}' f_0).$$

Existence of the solution to the snapshot problem

Let us go into the existence of the solution to the snapshot problem.

$$(SP)\ \begin{cases} \left(\Delta_{S^n} - \left(\frac{n-1}{2}\right)^2 - M^2\right) u(x,t) = \frac{\partial^2}{\partial t^2}\, u(x,t), \\ u\big|_{t=0} = f_0,\ \ u\big|_{t=t_1} = f_1,\ \ u\big|_{t=t_2} = f_2. \end{cases}$$

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Let us write $f_0,\ f_1,\ f_2$ as

$$f_0 = \sum_{l\geq 0} \sum_{m=1}^{d(l)} a_{lm}\, Y_{lm}, \quad f_1 = \sum_{l\geq 0} \sum_{m=1}^{d(l)} b_{lm}\, Y_{lm}, \quad f_2 = \sum_{l\geq 0} \sum_{m=1}^{d(l)} c_{lm}\, Y_{lm}.$$

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$$\widetilde{b_{lm}} \sin \lambda_M(l) t_2 = \widetilde{c_{lm}} \sin \lambda_M(l) t_1.$$

Here we introduce the following condition.

$$\text{(D)}\quad \left\{ \begin{array}{l} \quad \text{There exist positive constants } C \text{ and } \rho \text{ such that} \\ |\sin \lambda_M(l) t_1| + |\sin \lambda_M(l) t_2| \geq C(1+l)^{-\rho}, \quad \text{for } \forall\, l \geq 0. \end{array} \right.$$

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By the assumption that $f_0, f_1, f_2 \in C^\infty(S^n)$, the Fourier coefficients $\{a_{lm}\}$, $\{b_{lm}\}$, $\{c_{lm}\}$ of f_0, f_1, f_2 are rapidly decreasing respectively as $l \to \infty$, and so are $\{\widetilde{b_{lm}}\}$ and $\{\widetilde{c_{lm}}\}$.

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$$\begin{cases} f_1 & = S'_{t_1} f_0 + S_{t_1} g_0, \\ f_2 & = S'_{t_2} f_0 + S_{t_2} g_0. \end{cases}$$

Therefore, the solution u to the Cauchy problem

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Is the Diophantine condition (D) necessary and sufficient for the existence of the solution to the snapshot problem?

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We have the following.

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has a unique solution if and only if the following Diophantine condition (D) *holds.*

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Main theorem

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We assume that the three functions $f_0, f_1, f_2 \in C^\infty(S^n)$ *satisfy the compatibility condition* $S_{t_2}(f_1 - S'_{t_1} f_0) = S_{t_1}(f_2 - S'_{t_2} f_0)$*.*
Then the snapshot problem

$$(SP) \begin{cases} \left(\Delta_{S^n} - \left(\frac{n-1}{2}\right)^2 - M^2\right) u(x,t) = \frac{\partial^2}{\partial t^2} u(x,t), \\ u|_{t=0} = f_0, \ \ u|_{t=t_1} = f_1, \ \ u|_{t=t_2} = f_2. \end{cases}$$

has a unique solution if and only if the following Diophantine condition (D) *holds.*

$$(\mathrm{D}) \begin{cases} \quad \textit{There exist positive constants } C \textit{ and } \rho \textit{ such that} \\ |\sin \lambda_M(l) t_1| + |\sin \lambda_M(l) t_2| \geq C(1+l)^{-\rho}, \quad \textit{for } \forall\ l \geq 0. \end{cases}$$

We note that if (D) holds then $|\sin \lambda_M(l) t_1| + |\sin \lambda_M(l) t_2| > 0$ for all $l \geq 0$. Thus we have the uniqueness of the solution.

Outline of the proof of the “only if” part.

We briefly explain how to prove that the snapshot problem has a unique solution only if the Diophantine condition (D) holds.

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If the snapshot problem (SP) has a unique solution, the above map $\{f_0,\ g_0\} \mapsto \{f_0,\ f_1,\ f_2\}$ is continuous and bijective. By the open mapping theorem, the inverse map $\{f_0,\ f_1,\ f_2\} \mapsto \{f_0,\ g_0\}$ is also continuous,

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If the snapshot problem (SP) has a unique solution, the above map $\{f_0,\ g_0\} \mapsto \{f_0,\ f_1,\ f_2\}$ is continuous and bijective. By the open mapping theorem, the inverse map $\{f_0,\ f_1,\ f_2\} \mapsto \{f_0,\ g_0\}$ is also continuous, from which it follows that the Diophantine condition (D) holds.

What happens if (D) does not hold?

If the Diophantine condition (D) does not hold, there exist $f_0,\ f_1$ and f_2 satisfying the compatibility condition such that the snapshot problem with the snapshot data $\{f_0,\ f_1,\ f_2\}$ does not have a solution.

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More precisely, if (D) does not hold, the image of the mapping $\{f_0,\ g_0\} \mapsto \{f_0,\ f_1,\ f_2\}$ is a proper subspace of the space of snapshot data with the compatibility condition.

Two sufficient conditions

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1. Sufficient condition by a non-Liouville number

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1. Sufficient condition by a non-Liouville number
2. Sufficient condition by a badly approximable number

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From now on, we will explain the definition and the properties of a Liouville number.

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The following is a famous example given by Liouville.

$$x = \sum_{m=1}^{\infty} 10^{-m!}$$

Properties of Liouville numbers

Let $\mathcal{L}$ be the set of Liouville numbers. The following are known in number theory.

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As we mentioned, the set of Liouville numbers is a measure zero set. So the above theorem asserts that for almost all t_1 and t_2 the snapshot problem (SP) has a unique solution.

Sufficient condition by a badly approximable number

The above theorem asserts that the snapshot problem (SP) has a unique solution if t_1/t_2 is neither rational nor a Liouville number. So one may think of the following question.

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For simplicity, let us consider the case when t_1/t_2 is irrational.

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For example, the largest and the second largest Markov constants are given respectively by

$$m\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}, \qquad m(1+\sqrt{2}) = \frac{1}{2\sqrt{2}}.$$

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The Diophantine condition (D) follows from the above inequality.

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Then, the following theorem is well known in number theory.

Theorem (Steuding "Diophantine Analysis," Theorem 5.9)

x is badly approximable if and only if its partial quotients a_n $(n = 0, 1, 2, 3, \cdots)$ are bounded.

Examples of continued fraction expansions

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} \qquad \text{(the golden ratio)}$$

$$\sqrt{3} = [1; \overline{1, 2}] = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}$$

Some Remarks on M_0

As we stated, if $0 < M < M_0$, and if $\{f_0, f_1, f_2\}$ satisfies the compatibility condition, then the snapshot problem (SP) with the snapshot data $\{f_0, f_1, f_2\}$ has a unique solution.

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As we stated, if $0 < M < M_0$, and if $\{f_0, f_1, f_2\}$ satisfies the compatibility condition, then the snapshot problem (SP) with the snapshot data $\{f_0, f_1, f_2\}$ has a unique solution. In addition, M_0 is given explicitly by

$$M_0 := \max\left\{\sqrt{\frac{2\pi m(t_1/\pi)}{t_1}}, \sqrt{\frac{2\pi m(t_2/\pi)}{t_2}}\right\}.$$

where $m(t_j/\pi)$ is the Markov constant of $t_j/\pi,\ (j = 1.2)$.

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for all rationals p/q. The Markov constant $m(t_1/\pi)$ gives the upper bound of such c. More precisely, we have the following.

For any $\varepsilon > 0$, we have

$$\left| \frac{t_1}{\pi} - \frac{p}{q} \right| \geq \frac{m(t_1/\pi) - \varepsilon}{q^2},$$

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In fact, the above inequality gives the key estimate

$$\begin{aligned} & |\sin \lambda_M(l) t_1| + |\sin \lambda_M(l) t_2| \\ & \geq \frac{2}{3\pi} \times \frac{t_1^2 (M_0^2 - M^2)}{(2l + n - 1 + 2M_0) t_1 + \frac{\pi}{2}}, \quad \text{for } l \geq l_0. \end{aligned}$$

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If $m(t_j/\pi),\ (j = 1, 2)$ are small or $t_j/\pi,\ (j = 1, 2)$ are large, then M_0 becomes small.

For any $\varepsilon > 0$, we have

$$\left| \frac{t_1}{\pi} - \frac{p}{q} \right| \geq \frac{m(t_1/\pi) - \varepsilon}{q^2},$$

except for finitely many rationals p/q.

In fact, the above inequality gives the key estimate

$$\begin{aligned} & |\sin \lambda_M(l) t_1| + |\sin \lambda_M(l) t_2| \\ & \geq \frac{2}{3\pi} \times \frac{t_1^2 (M_0^2 - M^2)}{(2l + n - 1 + 2M_0) t_1 + \frac{\pi}{2}}, \quad \text{for } l \geq l_0. \end{aligned}$$

If $m(t_j/\pi),\ (j = 1, 2)$ are small or $t_j/\pi,\ (j = 1, 2)$ are large, then M_0 becomes small. In such a case, we can observe the motion of only a light particle.

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Thank you very much for your attention!

Remark on a 2-snapshot problem

In our snapshot problem

$$(SP)\ \begin{cases} \left(\Delta_{S^n} - \left(\frac{n-1}{2}\right)^2 - M^2\right) u(x,t) = \partial_t^2 u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{S}^n, \\ u|_{t=0} = f_0,\ \ u|_{t=t_1} = f_1,\ \ u|_{t=t_2} = f_2, \end{cases}$$

we give three functions $f_0,\ f_1$ and f_2 at three different times $t = 0, t_1$ and t_2 in order to get a unique solution. In fact, fot almost all $t_1,\ t_2$ (unless t_1/t_2 is a Liouville number), we get a unique solution.

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But in the case when $M = 0$, it makes sense to consider the following 2-snapshot problem.

$$\mathrm{(SP2)}\ \begin{cases} \left(\Delta_{\mathbb{S}^n} - \left(\frac{n-1}{2}\right)^2\right) u(t,x) = \partial_t^2 u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{S}^n, \\ u|_{t=0} = f_0,\ u|_{t=\alpha} = f_\alpha. \end{cases}$$

We call the above equation the shifted wave equation on $\mathbb{S}^n$.

In the above (SP2), we give two functions f_0 and f_α at two different times $t = 0$ and $t = \alpha$.

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Theorem (Christensen-Gonzalez-K-Wang, Adv. Math., (2024))

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The 2-snapshot problem (SP2) with the snapshot data $\{f_0,\ f_\alpha\}$ has a unique solution if and only if α/π is irrational and not a Liouville number.

In particular, for almost all α, (SP2) has a unique solution.