Classification of unstable travelling wave solutions to KdV type equations

Kaito Kokubu
Tokyo University of Science, D3

March 4, 2026　　Himeji conference on PDEs 2026

KdV type eq.

(fKdV) $$\partial_t u + \partial_x f(u) - \partial_x D_x^\sigma u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}$$

- $u = u(t,x)$: real-valued unknown function
- $1 \leq \sigma \leq 2$, $D_x^\sigma := \mathcal{F}^{-1}|\xi|^\sigma \mathcal{F} = (-\partial_x^2)^{\sigma/2}$ ($\mathcal{F}$: the Fourier transform)
  $\sigma = 2$ ⋯ KdV equation, $\sigma = 1$ ⋯ Benjamin–Ono equation

Aim of this study

To observe phenomena of travelling wave solutions to (fKdV) with nonlinearities $f(u) = u^p + u^q$, where p, $q \in \mathbb{N}$, $2 \leq p < q < \infty$.
Especially, to classify of the phenomena by the parity of indices p and q.

Example $\sigma = 2$, $p = 2$, $q = 3$ ⋯ the Gardner equation
(deduced by Miura–Gardner–Kruskal (1968))

Remark We can also consider nonlinearities of the form $f(u) = -u^p + u^q$.

(fKdV) $$\partial_t u + \partial_x(u^p + u^q) - \partial_x D_x^\sigma u = 0$$

Notation $H^{\sigma/2}$: L^2-based Sobolev space

Assumption (solvability of (fKdV))

$\forall u_0 \in H^{\sigma/2}$, $\exists T = T(\|u_0\|_{H^{\sigma/2}}) > 0$, $\exists! u(t) \in C([0,T), H^{\sigma/2})$
with conserved energy E and mass M;

$$E(v) = \frac{1}{2}\|D_x^{\sigma/2} v\|_{L^2}^2 - \frac{1}{p+1}\int_{\mathbb{R}} v^{p+1}\,dx - \frac{1}{q+1}\int_{\mathbb{R}} v^{q+1}\,dx,$$

$$M(v) = \frac{1}{2}\|v\|_{L^2}^2$$

Remark This assumption is satisfied if $\frac{4}{3} \le \sigma \le 2$. (Shown by Molinet–Tanaka (2025))

(fKdV) $$\partial_t u + \partial_x f(u) - \partial_x D_x^\sigma u = 0$$

<u>**Travelling wave solution**</u> $u(t,x) = \phi(x-ct)$ ($c > 0$: wave speed)

The function $\phi \in H^{\sigma/2}$ solves the following stationary problems:

(SP) $$D_x^\sigma \phi + c\phi - f(\phi) = 0, \quad x \in \mathbb{R}$$

<u>The action</u> $S_c(v) = E(v) + cM(v)$

where $E(v) = \frac{1}{2}\|D_x^{\sigma/2} v\|_{L^2}^2 - \int_{\mathbb{R}} F(u)\,dx$, $F(s) = \int_0^s f(s')\,ds'$, $M(v) = \frac{1}{2}\|v\|_{L^2}^2$

We can see that $S_c \in C^2(H^{\sigma/2}, \mathbb{R})$, $S_c'(v) = D_x^\sigma v + cv - f(v)$

<u>Definition</u> $\phi \in H^{\sigma/2}$ is a ground state of (SP)

$:\Longleftrightarrow$ $S_c(\phi) = \inf\{S_c(v) : v \in H^{\sigma/2} \setminus \{0\},$ v is a solution to (SP).$\}$

<u>Question 1</u> The existence of ground states to (SP).

(fKdV) $$\partial_t u + \partial_x f(u) - \partial_x D_x^\sigma u = 0$$

Stability and instability ⋯ Orbital stability

Definition

- $U_r(\phi) = \{v \in H^{\sigma/2} : \inf_{y\in\mathbb{R}} \|u - \phi(\cdot - y)\|_{H^{\sigma/2}} < r\}$, $\phi \in H^{\sigma/2}$, $r > 0$
- A travelling wave $\phi(x - ct)$ is **(orbitally) stable**
  $:\Longleftrightarrow$ $\forall\varepsilon > 0$ $\exists\delta > 0$ s.t.
  if $u_0 \in U_\delta(\phi)$, then the solution $u(t) \in C([0,\infty), H^{\sigma/2})$ to (fKdV) exists globally in time and satisfies $u(t) \in U_\varepsilon(\phi)$ for all $t \geq 0$.
- A travelling wave $\phi(x - ct)$ is **unstable** $:\Longleftrightarrow$ not stable in above sense.

Question 2 When travelling waves to (fKdV) are stable or unstable?

(fKdV) $\partial_t u + \partial_x f(u) - \partial_x D_x^\sigma u = 0$

(fKdV) with single power nonlinearities $f(u) = u^p$, $p \in \mathbb{N}$, $p \geq 2$.

- Stationary problem with single power nonlinearity
$$D_x^\sigma \psi + c\psi - \psi^p = 0$$
have a unique positive, even ground state ψ_c for each $c > 0$.
(Weinstein (1987), Amick–Toland (1991), Frank–Lenzmann (2013))

- Due to the abstract theory by Bona–Souganidis–Strauss (1987),
$p < 2\sigma + 1 \implies$ The travelling wave $\psi_c(x - ct)$ is stable for all $c > 0$.
$p > 2\sigma + 1 \implies$ The travelling wave $\psi_c(x - ct)$ is unstable for all $c > 0$.

- For $\sigma = 2$ and $p = 2\sigma + 1 = 5$ (L^2-critical KdV eq.)
The travelling wave $\psi_c(x - ct)$ is unstable for all $c > 0$. (Martel-Merle, 2001)

(fKdV) $$\partial_t u + \partial_x f(u) - \partial_x D_x^\sigma u = 0$$

(SP) $$D_x^\sigma \phi + c\phi - f(\phi) = 0$$

(fKdV) with double power nonlinearities (our problems)

$$f(u) = u^p + u^q \quad (p,\ q \in \mathbb{N},\ 2 \le p < q < \infty)$$

Considering non-odd nonlinearities, the existence of travelling waves to (fKdV) varies by the parity of indices p and q.

Existence of nontrivial solutions to (SP) (K., 2024, to appear)

q \ p	Even	Odd
Even	∃ Positive, even solution ∄ Negative solution	∃ Positive, even solution ∃ Negative, even solution if $\sigma = 1,\ c \ll 1$
Odd	∃ Positive, even **ground state** ∃ Negative, even solution	∃ Positive, even **ground state** ∃ Negative, even **ground state**

(fKdV) $\partial_t u + \partial_x f(u) - \partial_x D_x^\sigma u = 0$

(SP) $D_x^\sigma \phi + c\phi - f(\phi) = 0$

(fKdV) with double power nonlinearities (our problems)

$$f(u) = u^p + u^q \quad (p,\ q \in \mathbb{N},\ 2 \le p < q < \infty)$$

Stability of travelling wave solutions (K., to appear)

Assume q is odd, and let ϕ_c be a positive, even ground state to (SP) for $c > 0$.

(1) $p < 2\sigma + 1 \implies$ A travelling wave $\phi_c(x - ct)$ is stable for $c \ll 1$.
Example: $(p, q) = (2, 3)$ $\cdots$ the Gardner eq.,
$\sigma = 2$, $(p, q) = (3, 5)$ $\cdots$ q is L^2-critical.
and more...

(2) $q < 2\sigma + 1 \implies$ A travelling wave $\phi_c(x - ct)$ is stable for $c \gg 1$.
Example: $1 < \sigma \le 2$, $(p, q) = (2, 3)$ $\cdots$ the only case for (2)

(fKdV) $$\partial_t u + \partial_x f(u) - \partial_x D_x^\sigma u = 0$$

(SP) $$D_x^\sigma \phi + c\phi - f(\phi) = 0$$

Theorem (K., preprint)

Assume q is odd, and let ϕ_c be a positive, even ground state to (SP) for $c > 0$.

(1) $2\sigma + 1 \le p < q \implies$ A travelling wave $\phi_c(x - ct)$ is unstable for all $c > 0$.
Example: $\sigma = 2$, $(p, q) = (5, 7)$: p is L^2-critical.

(2) $p < 2\sigma + 1 < q \implies$ A travelling wave $\phi_c(x - ct)$ is unstable for $c \gg 1$.
Example: $1 < \sigma \le 2$, $p = 3$, q : odd

Remark We can also obtain the classification of unstable travelling waves for (fKdV) with $f(u) = -u^p + u^q$.

Sufficient condition ϕ_c : a ground state of (SP)

Proposition 1

Put $\phi_c^\lambda(x) = \lambda^{\frac{1}{2}}\phi_c(\lambda x)$. ($L^2$-norm invariant scaling)

If $\partial_\lambda^2 S_c(\phi_c^\lambda)|_{\lambda=1} < 0$, then a travelling wave $\phi_c(x-ct)$ is unstable.

With some identities of ϕ_c,

$$\partial_\lambda^2 S_c(\phi_c^\lambda)|_{\lambda=1} < 0 \iff \frac{\|\phi_c\|_{L^{p+1}}^{p+1}}{\|\phi_c\|_{L^{q+1}}^{q+1}} < \frac{q+1}{p+1}\cdot\frac{(q-1)\{q-(2\sigma+1)\}}{(p-1)(2\sigma+1-p)} \quad \cdots (\bigstar)$$

- $2\sigma+1 \le p < q \implies$ ($\bigstar$) holds for all $c > 0$.
- $p < 2\sigma+1 < q \implies$ ($\bigstar$) holds for $c \gg 1$ by approximation.

Lemma 1

ϕ_c : a positive ground state to (SP) for $c > 0$, $\breve{\phi}_c(x) = c^{-\frac{1}{q-1}}\phi_c(c^{-\frac{1}{\sigma}}x)$

$\implies$ $\breve{\phi}_c \to \psi_{1,q}$ in $H^{\sigma/2}$ as $c \to +\infty$,

where $\psi_{1,q}$ is the unique positive ground state of $D_x^\sigma\psi + \psi - \psi^q = 0$.

Hereafter, we assume $\partial_\lambda^2 S_c(\phi_c^\lambda)|_{\lambda=1} < 0$.

Strategy of proof

- Construct a Lyapunov type functional $J_A(v)$ which is bounded in a neighborhood of a ground state ϕ_c
- Under the contradiction assumption that a travelling wave $\phi_c(x - ct)$ is stable, find a solution $u(t)$ to (fKdV) that makes $|J_A(u(t))|$ diverges as $t \to +\infty$.
- The proof relies on the variational properties of ground states.

Lyapunov type functional

$$J_A(v) := \int_{\mathbb{R}} v(x+\tilde{z}(v))\Phi_A(x)\,dx, \quad v \in U_{\varepsilon_1}(\phi_c), \; A \geq 1$$

- $\Lambda\phi_c = \frac{1}{2}\phi_c + x\partial_x\phi_c \left(= \partial_\lambda \phi_c^\lambda|_{\lambda=1}\right), \quad \Phi(x) = \int_{-\infty}^{x} \Lambda\phi_c(y)\,dy$
- $\rho \in C_c^\infty(\mathbb{R}), \quad \rho_A(x) = \rho(x/A)$: a cut-off function, $\quad \Phi_A(x) = \Phi(x)\rho_A(x)$
- $\varepsilon_1 > 0, \quad \tilde{z} : U_{\varepsilon_1}(\phi_c) \to \mathbb{R}$: C^2-mapping s.t.

  $\tilde{z}(\phi_c) = 0, \; (v(\cdot + \tilde{z}(v)), \partial_x\phi_c)_{L^2} = 0$, and other properties

  (Modulation theory, cf. Bona–Souganidis–Strauss)

$\Longrightarrow \; |J_A(v)| \leq CA^{1/2}$ for $v \in U_{\varepsilon_1}(\phi_c)$

Time derivative of $J_A(u(t))$

$$\frac{d}{dt} J_A(u(t)) = -\langle S_c'(u(t)),\ \theta_A(u(t))\rangle_{H^{-\sigma/2},H^{\sigma/2}}$$

$$\theta_A(v) = \partial_x \Phi_A(\cdot - \tilde{z}(v)) + F_A(v)\partial_x \tilde{z}'(v), \quad F_A(v) = -\int_{\mathbb{R}} v(x)\partial_x \Phi_A(x - \tilde{z}(v))\,dx$$

Lemma 2

Put $P_A(v) = \langle S_c'(v), \theta_A(v)\rangle_{H^{-\sigma/2},H^{\sigma/2}}$.

Then, $\exists A_0 \geq 1$, $\exists \mu_0 > 0$, $\exists \varepsilon_0 \in (0, \varepsilon_1]$ s.t. $S_c(\phi_c) \leq S_c(v) + \mu_0 |P_{A_0}(v)|$

for all $v \in U_{\varepsilon_0}(\phi_c)$.

Key to the proof of Lemma 2

- Variational characterization of ground states

  ϕ_c : a ground state to (SP)

  $$\iff S_c(\phi_c) = \inf\{S_c(v) : v \in H^{\sigma/2} \setminus \{0\},\ K_c(v) = 0\}$$

  where $K_c(v) = \langle S_c'(v), v\rangle_{H^{-\sigma/2},H^{\sigma/2}}$: the Nehari Functional

- Properties of $\Lambda\phi_c$

  $$(\phi, \Lambda\phi_c)_{L^2} = 0, \quad \langle S_c''(\phi_c)\Lambda\phi_c, \Lambda\phi_c\rangle_{H^{-\sigma/2},H^{\sigma/2}} < 0$$

- Convergence of $\theta_A(\phi_c)$

  $$\theta_A(\phi_c) \to \Lambda\phi_c \text{ in } H^{\sigma/2} \text{ as } A \to +\infty.$$

  Here,

  $$\theta_A(\phi_c) = \partial_x\Phi_A + F_A(\phi_c)\partial_x\tilde{z}'(\phi_c), \quad F_A(\phi_c) = -\int_{\mathbb{R}} \phi_c(x)\partial_x\Phi_A(x)\,dx$$

Proof of Proposition 1

Suppose that a travelling wave $\phi_c(x-ct)$ is stable.

- $S_c'(\phi_c)=0,\ \partial_\lambda^2 S_c(\phi_c^\lambda)|_{\lambda=1}<0 \implies \exists\mu_1>0$ s.t. $S_c(\phi_c^{1-\mu_1})<S_c(\phi_c)$
- By contradiction assumption,
  $\exists u^{1-\mu_1}(t)\in C([0,\infty),H^{\sigma/2})$ with $u^{1-\mu_1}(0)=\phi_c^{1-\mu_1}$,
  $u^{1-\mu_1}(t)\in U_{\varepsilon_0}(\phi_c)$ for all $t\geq 0$.
- By Lemma 2 and the conservation laws,
  $\delta_{\mu_1}:=S_c(\phi_c)-S_c(\phi_c^{1-\mu_1})\leq \mu_0|P_{A_0}(u^{1-\mu_1}(t))|$,

  i.e. $$\left|\frac{d}{dt}J_{A_0}(u^{1-\mu_1}(t))\right|\geq\frac{\delta_{\mu_1}}{\mu_0}>0$$

$\implies \lim_{t\to+\infty}|J_{A_0}(u^{1-\mu_1}(t))|=+\infty$, which contradicts the boundedness of J_{A_0} in $U_{\varepsilon_0}(\phi_c)$.

Summary and Future works

Summary

- We observe conditions where a travelling waves to (fKdV) with $f(u) = u^p + u^q$ becomes unstable.
- The proof relies on the variational properties of ground states to (SP).

Future works

- Determine whether “nontrivial solutions” to (SP) is a ground state or not.
- Analyze the stability of a travelling wave constructed with an excited state of (SP).
- Observe (fKdV) with $f(u) = \pm u^p - u^q$.
  (The existence of ground states is difficult.)

Thank you for your attention!