Spectral flow for unitaries in $\mathrm{Id}+$Schatten

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March 3, 2026

Fredholm operators

Recall that a Fredholm operator $F \in B(H)$ is a bounded operator invertible modulo compact operators.

Example: the unilateral shift $S : e_n \mapsto e_{n+1}$ on $\ell^2(\mathbb{N})$.

The index of a Fredholm operator is

$$\mathrm{Index}(F) = \dim \ker(F) - \dim \ker(F^*).$$

Example: the index of the unilateral shift S is -1.

Self-adjoint Fredholm operators have zero index.

Spectral flow for self-adjoint Fredholm operators

Spectral flow for self-adjoint Fredholm operators was introduced by Lusztig and Atiyah-Patodi-Singer. For a norm continuous path $[0,1] \ni t \mapsto F_t$ their definition was

$$\begin{aligned}\mathrm{SF}(F_\bullet) &= \#\text{eigenvalues going from negative to positive} \\ &\quad - \#\text{eigenvalues going from positive to negative.}\end{aligned}$$

While intuitively simple and easy to compute in simple examples, the definition is difficult to make rigorous.

John Phillips gave the modern definition of spectral flow.

Phillips' definition

Lemma

Given two projections $P, Q \in B(H)$*, the operator* $PQ : QH \to PH$ *is Fredholm if and only if* $\|\pi(P) - \pi(Q)\| < 1$*, where* $\pi : B(H) \to B(H)/K(H)$ *is the quotient map. [1, Lemma 4.1]*

Given a norm continuous path $[0,1] \ni t \mapsto F_t$ of self-adjoint Fredholm operators, let $P_t := \chi_{\geq 0}(F_t)$ be the non-negative spectral projections. Typically, $t \mapsto P_t$ is norm discontinuous, but $t \mapsto \pi(P_t)$ is norm continuous in the Calkin algebra.

M-T. Benemeur, A. Carey, J. Phillips, A. Rennie, F. Sukochev, K. Wojciechowski, *An analytic approach to spectral flow in von Neumann algebras*, in 'Analysis, Geometry and Topology of Elliptic Operators-Papers in Honour of K. P. Wojciechowski', World Scientific, 2006, 297–352.

Phillips' definition

Definition

Given a norm continuous path $[0,1] \ni t \mapsto F_t$ of self-adjoint Fredholm operators, partition $[0,1]$ via $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $\|\pi(P_t) - \pi(P_s)\| < 1/2$ for all $t, s \in [t_{i-1}, t_i]$ and each i. Then

$$SF(F_\bullet) := \sum_{i=1}^{n} \mathrm{Index}(P_{i-1}P_i).$$

The spectral flow is independent of the partition and invariant under homotopies of the path keeping endpoints fixed.

The range of applications in operator algebras, global analysis and topology, and quantum physics is enormous.

A class of unitaries

In [1, 2] an analogous theory for certain paths of unitaries was developed. The reasons they did this will come back later in the talk.

The class of unitaries they considered was

$$ {}_F\mathcal{U}_I = \{U \in \mathrm{Id} + K : \; U + \mathrm{Id} \text{ is Fredholm}, \; U - \mathrm{Id} \text{ is injective}\}. $$

So -1 is an isolated eigenvalue of finite multiplicity and 1 is not an eigenvalue.

B. Booß-Bavnbek, M. Lesch, J. Phillips, *Unbounded Fredholm operators and spectral flow*, Canad. J. Math., **57** (2), 2005, 225–250.

P. Kirk, M. Lesch, *The η-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math., **16** (4), 2004, 553–629.

Spectral flow for unitaries

There is more subtlety to the definition of spectral flow for paths of unitaries $[0,1] \ni t \mapsto U_t$. Again we partition the interval $0 = t_0 < t_1 < \cdots < t_n = 1$, and then one must prove that there exist $\epsilon_j > 0$ such that

$$\ker(U_t - e^{i(\pi \pm \epsilon_j)}) = \{0\} \text{ for all } t_{j-1} \leq t \leq t_j.$$

Then

$$SF(U_\bullet) = \sum_{j=1}^{n} \Big(\sum_{0 \leq \theta < \epsilon_j} \dim\ker(U_{t_j} - e^{i(\pi+\theta)}) - \dim\ker(U_{t_{j-1}} - e^{i(\pi+\theta)}) \Big)$$

is independent of the partition and the choice of ϵ_j satisfying the requirements, and is a homotopy invariant.

The winding number

There are homotopy equivalences

$$
\begin{aligned}
{}_F\mathcal{U}_I &\sim \mathcal{U}_\infty = \{U \in \mathrm{Id} + K\} \\
&\sim \mathcal{U}_p = \{U \in \mathrm{Id} + \mathcal{L}^p\} \qquad \mathcal{L}^p = \{T \in K : \mathrm{Tr}(|T|^p) < \infty\} \\
&\sim \mathcal{U}_1 = \{U \in \mathrm{Id} + \mathcal{L}^1\}.
\end{aligned}
$$

Spectral flow is thus **defined** for paths in any of these spaces. For **loops** in $\mathcal{U}_1$ there is a formula that computes spectral flow:

$$SF(U_\bullet) = \frac{1}{2\pi i}\int_a^b \mathrm{Tr}(U_t^* \dot{U}_t)\, dt = \text{winding number}.$$

How, and why, should we extend the winding number formula to $\mathcal{U}_p$?

Motivation from scattering theory

For $H_0 = -\sum_{j=1}^{n} \partial_j^2$ on $\mathbb{R}^n$ and $H = H_0 + V$ with $V \in C_c^\infty(\mathbb{R}^n)$ (say), we can define the (isometric) wave operators and (unitary) scattering operator

$$W_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \qquad S = W_+^* W_-.$$

As $SH_0 = H_0 S$, in the spectral representation of H_0 we can regard S as a function of $\lambda \in (0, \infty)$ and

$$S(\lambda) \in \mathrm{Id}_{L^2(\mathbb{S}^{n-1})} + \mathcal{L}^1(L^2(\mathbb{S}^{n-1})).$$

Motivation from scattering theory

While $S(\lambda) - \mathrm{Id}$ is pointwise trace class, for the "integrated trace"

$$\tau(A(\cdot)) = \int_0^\infty \mathrm{Tr}_{L^2(\mathbb{S}^{n-1})}(A(\lambda))\, d\lambda$$

we only have $\|S(\lambda) - \mathrm{Id}\|_p \leq C\lambda^{-1/2+(n-1)/2p}$ for some $p > 1$, [1].

D. R. Yafaev. *Mathematical scattering theory: Analytic theory*, **158**, Mathematical surveys and monographs, American Mathematical Society, 2010.

Motivation from scattering theory

With some modern refinements [1, 2] Levinson's theorem states that

$$\mathrm{Index}(W_-) = -\#\text{bound states of } H = SF(S(\bullet)).$$

The last equality holds in the absence of zero energy resonances.

So finding an analytic formula for the spectral flow of Schatten unitaries will give an analytic expression for the number of bound states.

- J. Kellendonk, S. Richard. *On the wave operators and Levinson's theorem for potential scattering in* $\mathbb{R}^3$, Asian-Eur. J. Math., **5** (1), 2012.

- A. Alexander, A. Rennie. *Levinson's theorem as an index pairing*, J. Funct. Anal., **286** (5), 2024.

The set-up

The aim is to find an analytic formula for the spectral flow of loops of unitaries from Id to Id in $\mathcal{U}_p$. Focus on loops for simplicity (no eta invariants).

This is sufficient for application to the scattering matrix when there are no resonances, since then $S(0) = \mathrm{Id}$ and $\lim_{\lambda \to \infty} S(\lambda) = \mathrm{Id}$ in operator norm.

The main fact we rely on is that each $\mathcal{U}_p$ is a Banach manifold, and $\exp_{\mathrm{Id}} : \text{skew-}\mathcal{L}^p \to \mathcal{U}_p$ is surjective [1].

P. de la Harpe, *Classical Banach-Lie groups and Banach-Lie algebras of operators in Hilbert space*, PhD Thesis, University of Warwick, 1972.

What would Carey-Phillips do?

Carey-Phillips [1] obtained analytic formulae for paths of self-adjoint Fredholm operators satisfying Schatten conditions $F_t - F_0 \in \mathcal{L}^p$.

They achieved this by considering the affine space $F_0 + \mathcal{L}^p$, and showing that the spectral flow was the integral of an exact differential form on the infinite dimensional Banach manifold $F_0 + \mathcal{L}^p$.

The differential form they considered was

$$(F, X) \mapsto \mathrm{Tr}(X(1 - F^2)^n)$$

where $X \in \mathcal{L}^p$ is a tangent vector at the point F and $n > p - 1$.

A. Carey, J. Phillips. *Unbounded Fredholm modules and spectral flow*, Canad. J. Math. **50** (4), 673–718 (1998)

Guessing the differential form

The Banach manifold $\mathcal{U}_p$ is not flat like the affine space considered by Carey-Phillips. It does have a surjective exponential map from the tangent space at Id, so the task is not impossible.

We still need a guess for a differential form to start with.

This was provided by Kellendonk and Richard in [1]. Our aim is to turn this ansatz into a precise statement à la Carey-Phillips.

J. Kellendonk, S. Richard. *On the wave operators and Levinson's theorem for potential scattering in* $\mathbb{R}^3$, Asian-Eur. J. Math., **5** (1), 2012.

Exact one-forms

Theorem

Let n be an integer with $n > p - 1$ and r real with $r > (p-1)/2$. We define one-forms on tangent vectors X in the tangent space at $U \in \mathcal{U}_p$ by

$$\alpha_{n,U}(X) := (-1)^n \frac{1}{2\pi i} \mathrm{Tr}(X(U-1)^n) \qquad \textit{K-R differential form}$$

$$\beta_{r,U}(X) := -i \frac{\Gamma(2r+1)}{\sqrt{\pi}\Gamma(2r+1/2)} \left(\frac{1}{2}\right)^{2r+1} \mathrm{Tr}(X|U - \mathrm{Id}|^{2r}).$$

Then $\alpha_{n,U}$ and $\beta_{r,U}$ are exact differential forms.

Exact one-forms

To prove that α, β are closed forms is easy. One need only apply the definition of the exterior derivative.

To prove that they are exact requires us to show that they are differentials of functions on $\mathcal{U}_p$.

Given $U \in \mathcal{U}_p$ with $U = e^Y$, $Y \in \mathcal{L}^p$ skew-adjoint, let $U_t = e^{tY}$. Then define

$$A_n(U) = (-1)^n \frac{1}{2\pi i} \int_0^1 \mathrm{Tr}(Y(U_t - 1)^n)\, dt$$

$$B_r(U) = -i \frac{\Gamma(2r+1)}{\sqrt{\pi}\Gamma(2r+1/2)} \left(\frac{1}{2}\right)^{2r+1} \int_0^1 \mathrm{Tr}(Y|U_t - 1|^{2r})\, dt.$$

Then $dA_n = \alpha_n$ and $dB_r = \beta_r$.

The spectral flow formula

Theorem

For a loop $U_\bullet \in \mathcal{U}_p$

$$SF(U_\bullet) = (-1)^n \frac{1}{2\pi i} \int_0^1 \mathrm{Tr}(U_t^* \dot{U}_t (U_t - 1)^n)\, dt$$
$$= -i \frac{\Gamma(2r+1)}{\sqrt{\pi}\,\Gamma(2r+1/2)} \left(\frac{1}{2}\right)^{2r+1} \int_0^1 \mathrm{Tr}(U_t^* \dot{U}_t |U_t - 1|^{2r})\, dt.$$

Homotopy invariance

To prove homotopy invariance, consider the Banach manifold $Map(S^1,\mathcal{U}_p)$ with tangent bundle $TMap = Map(S^1,\mathcal{L}^p)$. Then for a loop $U_\bullet \in Map(S^1,\mathcal{U}_p)$ we can apply the differential to the function $H_n(U_\bullet) = const \int_0^1 \alpha_{n,U_t}(U_t^* \dot{U}_t)\, dt$ to find

$$(d_{Map}H_n)(U_\bullet) = const \int_0^1 d\alpha_{n,U_t}(U_t^* \dot{U}_t)\, dt = 0.$$

So α_n is locally constant.

Then one checks the normalisation on simple paths and invokes [1] to conclude the theorem.

M. Lesch, *The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators*, in Spectral geometry of manifolds with boundary and decomposition of manifolds, Contemp. Math., **366**, Amer. Math. Soc., Providence, RI, 2005, 193–224.

The application to scattering

Applying the theory of determinants on Schatten ideals and analysis of high energy behaviour [1], we find

Theorem

In the absence of zero energy resonances

$$-\#\textit{bound states} = SF(S(\bullet)) = \frac{1}{2\pi i}\int_0^\infty \left(\mathrm{Tr}(S^*\dot{S})(\lambda) - p_n(\lambda)\right) d\lambda - \frac{1}{2\pi i}P_n(0)$$

where $P_n(\lambda) = \sum_{j=1}^{[n/2]} c_j(n,V)\lambda^{n/2-j}$ *and* $p_n = P_n'$.

Eg: $P_2(\lambda) = c\int_{\mathbb{R}^n} V(x)\,d^n x$, $\quad P_3(\lambda) = \lambda^{1/2}\tilde{c}\int_{\mathbb{R}^n} V(x)\,d^n x$,
$P_4(\lambda) = c_1\lambda\int_{\mathbb{R}^n} V(x)\,d^n x + c_2\int_{\mathbb{R}^n} V(x)^2\,d^n x$

A. Alexander. *Trace formula and Levinson's theorem in the presence of resonances*, Rev. Math. Phys., 10.1142/S0129055X24500363, 2024.

Spectral flow for unbounded operators

If $t \mapsto D_t$ is a path of unbounded self-adjoint Fredholm operators, then one can define spectral flow in several different ways. The different ways reflect different topologies on the space of unbounded operators, [1, 2].

1) Riesz continuous paths: $SF(D_\bullet) := SF(D_\bullet(1 + D_\bullet^2)^{-1/2})$.

2) Gap continuous paths: $SF(D_\bullet) := SF((D_\bullet - i)(D_\bullet + i)^{-1})$.

3) Wahl continuous paths: not discussed here.

B. Booß-Bavnbek, M. Lesch, J. Phillips, *Unbounded Fredholm operators and spectral flow*, Canad. J. Math., **57** (2), 2005, 225–250.

C. Wahl. *A new topology on the space of unbounded selfadjoint operators, K-theory and spectral flow*. In C^*-algebras and elliptic theory II, 297–309, Trends Math., Birkhäuser, Basel (2008).

Loosening the choices

The Riesz topology on operators on a Hilbert space H is defined by the norm continuity of the Riesz transform $D \mapsto D(1+D^2)^{-1/2}$. Spectral flow in this topology has been widely studied.

The gap topology is defined by norm continuity of the graph projections $P_{G(D_t)} : H \oplus H \to H \oplus H$. It is equivalent to norm resolvent continuity and the Cayley transform $D \to (D-i)(D+i)^{-1}$ is continuous in this topology.

The range of the Cayley transform on self-adjoint unbounded Fredholm operators is precisely ${}_F\mathcal{U}_I$, which is why this space of unitaries arose.

Moving Hilbert spaces

The inverse Cayley transform

$$U \mapsto i(U+1)(U-1)^{-1}$$

is well-defined on all unitaries and gives an unbounded self-adjoint operator defined on the domain

$$(U-1)H \quad \text{which is dense in the Hilbert space} \quad \overline{(U-1)H} \subset H.$$

The closure $\overline{(U-1)H}$ may not remain constant along a path of unitaries.

Moving Hilbert spaces

For $V \subset H$ a closed subspace and $1 \leq p \leq \infty$ define $\mathcal{F}_p(V)$ to be

$\{T : \mathrm{Dom}(T) \to V : \; T$ densely-defined, self-adjoint , $(T \pm i)^{-1} \in \mathcal{L}^p(V)\}$

and

$$\mathcal{F}_p(H) = \bigcup_{V \subset H} \mathcal{F}_p(V)$$

For $p = \infty$ instead of $\mathcal{L}^p$ we use the compacts K.

Moving Hilbert spaces

Proposition

The inverse Cayley transform $U \mapsto i(U+1)(U-1)^{-1}$ gives a bijection

$$\mathcal{U}_p(H) \to \mathcal{F}_p(H)$$

with inverse

$$T \mapsto \begin{cases} (T-i)(T+i)^{-1} & \text{on } V \\ \mathrm{Id} & \text{on } V^\perp \end{cases}.$$

So $\mathcal{F}_p(H)$ is a Banach manifold.

Observe that $\mathcal{U}_p \ni \mathrm{Id} \mapsto 0 : \{0\} \to \{0\}$.

Exact forms for moving Hilbert spaces

Theorem

Given $U \in \mathcal{U}_p(H)$ *set* $H_U = \overline{(U-1)H}$ *and* $C(U) = i(U+1)(U-1)^{-1}$. *For* $X \in \mathcal{L}^p$ *and* $n > p-1$ *define*

$$\alpha_{C(U),n}(X) = \frac{1}{2i}\mathrm{Tr}_{H_U}(X(C(U)-i)^{-n})$$

and for $r > (p-1)/2$ *define*

$$\beta_{C(U),r}(X) = -\frac{\Gamma(2r+1)}{\sqrt{\pi}\Gamma(2r+1/2)}\mathrm{Tr}_{H_U}(X|C(U)-i|^{-2r}).$$

Then both α_n, β_r *are exact forms on* $\mathcal{F}_p(H)$.

As a result we obtain spectral flow formulae for paths of unbounded operators on moving Hilbert spaces.

Continuity for operators on moving Hilbert spaces

The space $\mathcal{F}_p(H)$ inherits its topology and smooth structure from $\mathcal{U}_p(H)$. What does continuity mean?

Proposition

Let $T : \mathrm{Dom}(T) \subset V \to V$ with $V \subset H$ a closed subspace. Let $P_{G(T)} \in B(V \oplus V)$ be the graph projection of T.

Extend $P_{G(T)}$ to all of $H \oplus H$ by zero and define

$$P_T := P_{G(T)} + 0_H \oplus 1_{V^\perp}.$$

Then $t \mapsto T_t$ is continuous in $\mathcal{F}_p(H)$ if and only if $t \mapsto P_{T_t}$ is p-norm continuous. For $p = \infty$ this is norm continuity.

Disturbing: $0_H \oplus 1_{V^\perp}$ is not the graph projection of an operator.

Continuity for operators on moving Hilbert spaces

Example For U_t a path in $\mathcal{U}_p(H)$ the graph projection of $C(U_t)$ is

$$P_{G(C(U_t))} = \frac{1}{4}\begin{pmatrix} (U_t-1)(U_t^*-1) & i(U_t+1)(U_t^*-1) \\ i(U_t+1)(U_t^*-1) & 4P_{\overline{(U_t-1)H}} - (U_t-1)(U_t^*-1) \end{pmatrix}$$

and

$$P_{C(U_t)} = \frac{1}{4}\begin{pmatrix} (U_t-1)(U_t^*-1) & i(U_t+1)(U_t^*-1) \\ i(U_t+1)(U_t^*-1) & 4\mathrm{Id}_H - (U_t-1)(U_t^*-1) \end{pmatrix}.$$

Corollary

Let $t \to T_t \in \mathcal{F}_p(H)$ be a continuous path. The graph projections of $t \mapsto T_t$ are strongly continuous if and only if the projections $H \to H_t$ are strongly continuous.

Applications

These results encompass known results on spectral flow for gap continuous families of operators on fixed Hilbert spaces with moving domains. (Wahl, Booss-Bavnbek-Lesch-Phillips). Spectral flow for operators with varying domain is already important.

The next class of examples are those associated to continuous fields of Hilbert spaces. To stay connected to loops of unitaries we require the

Definition

Given $(T_t)_{t\in(-1,1)} \subset \mathcal{F}_\infty(H)$, extend the resolvents $(T_t \pm i)^{-1}$ from $H_t \subset H$ to all of H by zero. We say that $(T_t)_{t\in(-1,1)}$ vanishes at infinity if $\lim_{t\to\pm 1}(T_t \pm i)^{-1} = 0$ in norm. (Could use Schatten norm here too).

Continuous fields

Theorem

Let $(-1,1) \ni t \mapsto H_t \subset H$ be a continuous field of Hilbert spaces, and T_t a family of densely-defined self-adjoint operators on each H_t.

Then the set

$$\Gamma = \{\xi_\bullet : \xi_t \in \mathrm{Dom}(T_t) \text{ and } t \mapsto \langle \xi_t, \xi_t \rangle_t \text{ is continuous}\}$$

is dense in $\int_{(-1,1)}^{\oplus} H_t$ if and only if $t \mapsto T_t$ is a continuous path in $\mathcal{F}_\infty(H)$ and the graph projections $t \mapsto P_{G(T_t)}$ are strongly continuous.

Moreover, such a path has $t \mapsto (T_t \pm i)^{-1}$ in $C_0(K(H))$ provided (T_t) vanishes at infinity.

Kasparov classes for spectral flow

Theorem

Let $X_{C_0(\mathbb{R})}$ be a countably generated right Hilbert module, and T an unbounded self-adjoint regular operator on $X_{C_0(\mathbb{R})}$.
If $(\mathbb{C}, X_{C_0(\mathbb{R})}, T)$ is an unbounded Kasparov module defining a class in $KK^1(\mathbb{C}, C_0(\mathbb{R}))$, then it is unitarily equivalent to a cycle $(\mathbb{C}, P_\bullet(H \otimes C_0(\mathbb{R})), \tilde{T}_\bullet)$ with $t \mapsto P_t$ a strongly continuous path of projections on H and $t \mapsto \tilde{T}_t$ a continuous path in $\mathcal{F}^{sa}_\infty(H)$ vanishing at infinity.

Here $\tilde{T}_t$ is densely-defined on $P_t H$.

Conversely, every continuous path $t \mapsto T_t$ in $\mathcal{F}^{sa}_\infty(H)$ vanishing at infinity whose graph projections $P_{G(T_t)}$ are strongly continuous defines an unbounded Kasparov module $(\mathbb{C}, (H_\bullet)_{C_0(\mathbb{R})}, T_\bullet)$ and so a class in $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.
Here $H_t \subset H$ is the subspace on which T_t is densely-defined.

THANK YOU!