Energy decay of solutions to the wave equation with space-dependent effective damping localized near infinity

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偏微分方程式姫路研究集会

Slides available at
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Plan of talk

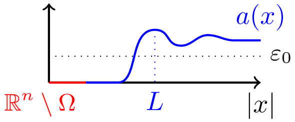
1. Introduction: damped wave equation & energy decay

$$u_{tt} - \Delta u + u_t = 0, \quad t > 0, x \in \mathbb{R}^n$$

2. Localized damping near infinity

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

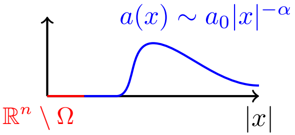
Results by Nakao (2001) & Ikehata (2003)



3. Effective damping localized near infinity

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

Main result (in preparation)



1. Introduction: damped wave equation & energy decay

$$u_{tt} - \Delta u + u_t = 0, \quad t > 0, x \in \mathbb{R}^n$$

Damped wave equation

Damped wave equation

Consider the initial value problem for the damped wave equation in the whole space $\mathbb{R}^n$:

(1) $$\begin{cases} u_{tt} - \Delta u + u_t = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

- $u = u(t,x)$: real-valued unknown function
- This equation describes the propagation of waves decaying by the effect of damping or friction.
- This equation is also called telegrapher's equation or dissipative wave equation, etc.

Energy

(2) $$E(t) := \frac{1}{2}\int_{\mathbb{R}^n} \left(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2\right)\,dx \quad (t \geq 0)$$

is called the (total) energy of u.

Proposition 1

If $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*, then the solution* u *to* (1) *satisfies*

$$E(t) + \int_0^t \int_{\mathbb{R}^n} |u_t(s,x)|^2\,dxds = E(0) \qquad (t \geq 0).$$

In particular, $E(t)$ *is non-increasing.*

Proof of Proposition 1

We formally calculate

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n}\left(|u_t|^2+|\nabla u|^2\right)\,dx\right)\\ &= \int_{\mathbb{R}^n}\left(u_t u_{tt}+\nabla u\cdot\nabla u_t\right)\,dx\\ &= \int_{\mathbb{R}^n}\left(u_t u_{tt}-u_t\Delta u\right)\,dx\\ &= -\int_{\mathbb{R}^n}|u_t|^2\,dx.\end{aligned}$$

Integrating the above on $[0,t]$, we get

$$E(t)+\int_0^t\int_{\mathbb{R}^n}|u_t(s,x)|^2\,dxds = E(0).$$

□

Sharp decay estimate of $E(t)$

Question (Energy decay problem)

$\lim_{t\to\infty} E(t) = 0$? If so, what is the sharp decay rate in t ?

In the whole space case (1), we can easily obtain the following answer.

Proposition 2

If $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then the solution u to (1) satisfies

$$E(t) \le C(1+t)^{-1} \left(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2 \right) \qquad (t \ge 0).$$

Moreover, the decay order $(1+t)^{-1}$ is sharp.

Proof of Proposition 2

There are several ways for the proof (Fourier analysis, solution formula, ...). But here we use the energy method for the later purpose.

We put

$$E_0(t) := \int_{\mathbb{R}^n} \left(u(t,x)u_t(t,x) + \frac{1}{2}|u(t,x)|^2 \right) dx$$

and formally calculate

$$\begin{aligned} \frac{d}{dt}E_0(t) &= \int_{\mathbb{R}^n} \left(u_t^2 + uu_{tt} + uu_t \right) dx \\ &= \int_{\mathbb{R}^n} \left(u_t^2 + u\Delta u \right) dx \\ &= \int_{\mathbb{R}^n} \left(u_t^2 - |\nabla u|^2 \right) dx. \end{aligned}$$

Then, we need to control the positive bad term u_t^2.

Combining $E(t)$ and $E_0(t)$, we define the modified energy by

$$\mathcal{E}(t) := 2(1+t)E(t) + E_0(t).$$

Then, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\left[2(1+t)E(t) + E_0(t)\right] \\
&= 2E(t) + 2(1+t)\frac{d}{dt}E(t) + \frac{d}{dt}E_0(t) \\
&= 2E(t) - 2(1+t)\int_{\mathbb{R}^n} u_t^2\,dx + \int_{\mathbb{R}^n}\left(u_t^2 - |\nabla u|^2\right)dx \\
&= \int_{\mathbb{R}^n}\left(u_t^2 + |\nabla u|^2 - 2(1+t)u_t^2 + u_t^2 - |\nabla u|^2\right)dx \\
&= -2t\int_{\mathbb{R}^n} u_t^2\,dx \\
&\le 0.
\end{aligned}$$

Integrating the preceding inequality on $[0, t]$, we get

$$\mathcal{E}(t) \leq \mathcal{E}(0) \qquad (t \geq 0).$$

On the other hand, by the Schwarz inequality, one can easily show

$$\begin{aligned}\mathcal{E}(t) &:= 2(1+t)\int_{\mathbb{R}^n}(|u_t|^2 + |\nabla u|^2)\,dx + \int_{\mathbb{R}^n}\left(uu_t + \frac{1}{2}|u|^2\right)\,dx \\ &\sim (1+t)E(t) + \|u(t)\|_{L^2}^2.\end{aligned}$$

Thus, we conclude

$$E(t) \leq C(1+t)^{-1}\left(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2\right) \qquad (t \geq 0).$$

For the sharpness of the order $(1+t)^{-1}$, we need to investigate the asymptotic profile of $u(t,x)$.

Actually, we can prove that

$$u(t) \sim e^{t\Delta}(u_0 + u_1) \qquad \text{for large } t$$

by using the Fourier analysis, solution formula, or energy method, etc. (we omit the details).

Moreover, the sharpness of the estimate

$$\|\partial_t^j \partial_x^\alpha e^{t\Delta} f\|_{L^p} \le C t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-j-\frac{|\alpha|}{2}} \|f\|_{L^q} \qquad (t > 0, f \in L^q(\mathbb{R}^n))$$

for $1 \le q \le p \le \infty$, $j \in \mathbb{Z}_{\ge 0}$, $\alpha \in \mathbb{Z}_{\ge 0}^n$ is well known (by, e.g., the scaling argument).

So, we have the sharpness of the order $(1+t)^{-1}$ in the estimate

$$E(t) \le C(1+t)^{-1}\left(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2\right) \qquad (t \ge 0).$$

□

Remark: L^p-L^q type decay

Like the L^p-L^q estimates for the heat equation

$$\|\partial_t^j \partial_x^\alpha e^{t\Delta} f\|_{L^p} \leq C t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-j-\frac{|\alpha|}{2}} \|f\|_{L^q} \qquad (t > 0, f \in L^q(\mathbb{R}^n))$$

for $1 \leq q \leq p \leq \infty$, $j \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{Z}^n_{\geq 0}$, we can get the following better decay estimate if we assume the additional decay for the initial data at spatial infinity.

Proposition 3 (Matsumura (1976))

$$E(t) \leq C(1+t)^{-n\left(\frac{1}{q}-\frac{1}{2}\right)-1} \left(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2 + \|u_0\|_{L^q}^2 + \|u_1\|_{L^q}^2\right) \qquad (t \geq 0)$$

for $1 \leq q \leq 2$.

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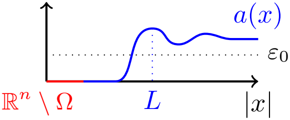
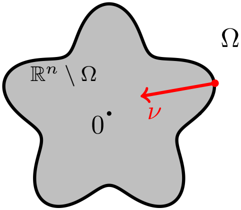
$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

Results by Nakao (2001) & Ikehata (2003)



Exterior domain Ω

- $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$: exterior domain ($\mathbb{R}^n \setminus \Omega$: compact)
- $\partial\Omega$: smooth
- $\mathbb{R}^n \setminus \Omega$: star-shaped with respect to $0 \notin \bar{\Omega}$
- ν : outward normal unit vector of $\partial\Omega$



Wave equation with space-dependent damping

Wave equation with space-dependent damping

Consider the initial-boundary value problem for the damped wave equation in Ω:

$$(3) \qquad \begin{cases} u_{tt} - \Delta u + a(x) u_t = 0, & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \Omega. \end{cases}$$

- $a(x) \in C^\infty(\overline{\Omega}) \cap L^\infty(\Omega)$, $a(x) \geq 0$
- $a(x)$ describes the strength of the damping at $x \in \Omega$.
- For $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the existence of unique solution can be proved by the Hille–Yosida theorem.

Energy

$$E(t) := \frac{1}{2}\int_{\Omega} \left(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2\right) dx \quad (t \geq 0).$$

In the same way as Proposition 1, we can show

$$E(t) + \int_0^t \int_{\Omega} a(x)|u_t(s,x)|^2 \, dxds = E(0) \qquad (t \geq 0).$$

In particular, $E(t)$ is non-increasing.

Question (Energy decay problem)

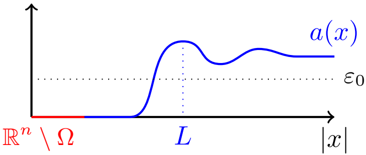
$\lim_{t\to\infty} E(t) = 0$? If so, how the decay rate in t depends on $a(x)$?

Results of Nakao (2001, Math. Z.) & Ikehata (2003, JDE)

Assumption 1

There exist $L > 0$ and $\varepsilon_0 > 0$ such that

$$a(x) \geq \varepsilon_0 \quad \text{for } |x| \geq L.$$



Nakao (2001, Math. Z.)

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then the solution u to (3) satisfies

$$E(t) \leq C(1+t)^{-1} \left(\|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right) \qquad (t \geq 0).$$

Ikehata (2003, JDE)

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and

$$I_0 := \int_\Omega |u_1(x) + a(x)u_0(x)|^2 d(x)^2 \, dx < +\infty,$$

where

$$d(x) := \begin{cases} |x| & (n \geq 3), \\ |x| \log(B|x|) & (n = 2) \end{cases}$$

with some $B \gg 1$, then the solution u to (3) satisfies

$$E(t) \leq C(1+t)^{-2} \left(\|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 + I_0 \right) \qquad (t \geq 0).$$

This result implies that additional decay assumption on initial data gives faster decay of energy.

Outline of Proof of Nakao (2001)

Lemma 4 (cf. Morawetz (1961), J. L. Lions (1988), Zuazua (1990))

Let $h(x) = (h_1(x), \dots, h_n(x)) \in \mathrm{Lip}(\overline{\Omega})$. *Then, we have*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_t h(x)\cdot\nabla_x u\,dx\\&= -\frac{1}{2}\int_\Omega (\nabla_x\cdot h)\left(|u_t|^2 - |\nabla_x u|^2\right)dx + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 \nu\cdot h\,dS\\&\quad -\int_\Omega \sum_{i,j=1}^n \partial_i u\partial_j u\partial_i h_j\,dx - \int_\Omega a(x)u_t(h\cdot\nabla_x u)\,dx.\end{aligned}$$

We want to take $h(x)$ so that

$\nabla_x\cdot h(x) > 0$ and $\sum_{i,j=1}^n \partial_i u\partial_j u\partial_i h_j > 0$ in $|x| < L$ and $\nu\cdot h \le 0$ on $\partial\Omega$.

To this end, we choose $h(x)$ as

$$h(x) := \phi(|x|)x$$

with

$$\phi(r) := \begin{cases} \varepsilon_0 & (r \leq L), \\ \dfrac{\varepsilon_0 L}{r} & (r > L). \end{cases}$$

Then, it is obvious that

$$\nabla_x \cdot h(x) = \varepsilon_0 n > 0 \quad \text{and} \quad \sum_{i,j=1}^{n} \partial_i u \partial_j u \partial_i h_j = \varepsilon_0 |\nabla_x u|^2 \quad \text{in} \quad |x| < L$$

Moreover, since $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to 0, we have $\nu \cdot h(x) = \varepsilon_0 \nu \cdot x \leq 0$ on $x \in \partial\Omega$.

Based on Lemma 4, we define the following modified energy.

Let $\mu, \lambda > 0$ and

$$\mathcal{E}(t) := \int_\Omega u_t h(x) \cdot \nabla_x u \, dx + \mu \widetilde{E_0}(t) + (1 + \lambda t) E(t).$$

Here, we put

$$\widetilde{E_0}(t) := \int_\Omega \left(u(t,x) u_t(t,x) + \frac{a(x)}{2} |u(t,x)|^2 \right) dx.$$

Then, we can obtain

$$\frac{d}{dt}\mathcal{E}(t) \le -\frac{n}{2}\varepsilon_0 \int_{\substack{\Omega \\ |x|<L}} |u_t|^2 \, dx - (1+\lambda t) \int_\Omega a(x) |u_t|^2 \, dx + (\mu + \lambda) \int_\Omega |u_t|^2 \, dx$$
$$- (\mu - \lambda + \varepsilon_0) \int_\Omega |\nabla_x u|^2 \, dx + \frac{n}{2}\varepsilon_0 \int_{\substack{\Omega \\ |x|<L}} |\nabla_x u|^2 \, dx + \text{(harmless terms)}.$$

Choosing μ, λ appropriately, we can show (RHS) ≤ 0. □

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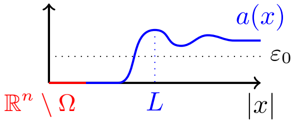
$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

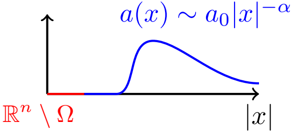
Results by Nakao (2001) & Ikehata (2003)



3. Effective damping localized near infinity

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

Main result (in preparation)



Space-dependent effective damping

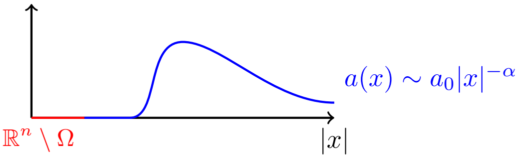
Let us change the assumption of Nakao & Ikehata to the following.

Assumption 2

$a \in C^\infty(\overline{\Omega})$, $a(x) \geq 0$, and there exist $\alpha \in [0,1)$ and $a_0 > 0$ such that

$$\lim_{|x|\to\infty} |x|^\alpha a(x) = a_0.$$

The damping term having the above behavior is so-called effective damping.



Main result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain with smooth boundary, and assume that $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to 0 $(\notin \overline{\Omega})$.
Consider the initial-boundary value problem of

$$
\begin{cases} u_{tt} - \Delta u + a(x) u_t = 0, & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{4}
$$

Theorem 5 (In preparation)

Under Assumption 2, if $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *satisfies*

$$
I_0 := \int_\Omega \left(|u_1(x)|^2 + |\nabla_x u_0(x)|^2 + |u_0(x)|^2 \langle x \rangle^{-\alpha} \right) \langle x \rangle^{\lambda(2-\alpha)} \, dx < +\infty
$$

with some $\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1})$, *where* $\langle x \rangle := \sqrt{1+|x|^2}$, *then the solution* u *to* (4) *satisfies*

$$
E(t) \leq C(1+t)^{-\lambda-1} I_0 \qquad (t \geq 0).
$$

Remarks (1)

- $\alpha = \lambda = 0 \;\Rightarrow\; E(t) \le C(1+t)^{-1}$.
  This corresponds to the result of Nakao (2001).
- $\alpha = 0 \;\Rightarrow\;$ we can choose λ from the interval $[0, \frac{n}{2} \cdot \frac{n}{2n-1})$.
  Then, we have the faster decay estimate $E(t) \le C(1+t)^{-\lambda-1}$ under the additional decay assumption $I_0 < +\infty$ on the initial data.
  - $n = 2 \;\Rightarrow\; \frac{n}{2} \cdot \frac{n}{2n-1} = \frac{2}{3} < 1$
  - $n = 3 \;\Rightarrow\; \frac{n}{2} \cdot \frac{n}{2n-1} = \frac{9}{10} < 1$
  - $n \ge 4 \;\Rightarrow\; \frac{n}{2} \cdot \frac{n}{2n-1} > 1$
    So, we do not cover Ikehata's (2005) conclusion $E(t) \le C(1+t)^{-2}$ for $n = 2, 3$.
    (The assumption on the initial data is also different.)

Remarks (2)

- For general $\alpha \in [0,1)$:
  If we strengthen Assumption 2 so that $a(x)$ is positive everywhere and $a(x) \sim a_0 |x|^{-\alpha}$ as $|x| \to \infty$, we have the following:

(Sobajima–W. (2021, JMSJ))

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies

$$I_0 := \int_\Omega \left(|u_1(x)|^2 + |\nabla_x u_0(x)|^2 + |u_0(x)|^2 \langle x \rangle^{-\alpha} \right) \langle x \rangle^{\lambda(2-\alpha)} \, dx < +\infty$$

with some $\lambda \in [0, \frac{n-\alpha}{2-\alpha})$, then the solution u to (4) satisfies

$$E(t) \le C(1+t)^{-\lambda-1} I_0 \qquad (t \ge 0).$$

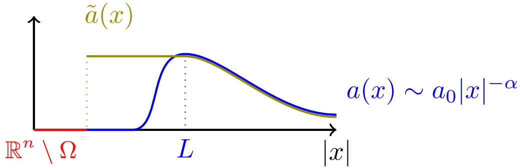
So, we believe that our restriction "$\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1})$" is technical and should be relaxed to "$\lambda \in [0, \frac{n-\alpha}{2-\alpha})$".

Outline of Proof

Let $L > 0$ be sufficiently large so that $a(x) \geq \frac{a_0}{2}|x|^{-\alpha}$ for $|x| > L$.

We define $\tilde{a}(x)$ so that

$$\tilde{a}(x) = a(x) \text{ in } |x| > L \qquad \text{and} \qquad \tilde{a}(x) > 0 \text{ everywhere.}$$



For $\tilde{a}(x)$, Sobajima–W. (2021, JMSJ) constructed a weight function $\Phi(t,x)$ such that

$$\Phi(t,x) \sim (t_0 + t + \langle x\rangle^{2-\alpha})^{-\beta}$$

and

$$\tilde{a}(x)\partial_t\Phi(t,x) - \Delta\Phi(t,x) \geq c\,\tilde{a}(x)\left(t_0 + t + \langle x\rangle^{2-\alpha}\right)^{-\beta-1},$$

where

$$\beta \in \left[0, \frac{n-\alpha}{2-\alpha}\right), \quad t_0 \geq 1$$

are parameters and c is a constant.

The above inequality means that Φ is a supersolution of the heat equation $\tilde{a}(x)\partial_t v - \Delta v = 0$.

By this weight function Φ, we define the weighted energies by

$$E_1(t) := \frac{1}{2}\int_\Omega \left(|u_t|^2 + |\nabla_x u|^2\right)\left(t_0 + t + \langle x\rangle^{2-\alpha}\right)^{\lambda + \frac{\alpha}{2-\alpha}}\, dx,$$

$$E_0(t) := \int_\Omega \left(u u_t + \frac{a(x)}{2}|u|^2\right)\Phi(t,x)^{-1+2\delta}\, dx,$$

where $\delta \in (0, 1/2)$ is a parameter.

These $E_1(t)$ and $E_0(t)$ satisfy the following good energy estimates.

$$E_1(t) := \frac{1}{2}\int_\Omega \left(|u_t|^2 + |\nabla_x u|^2\right)\left(t_0 + t + \langle x\rangle^{2-\alpha}\right)^{\lambda + \frac{\alpha}{2-\alpha}}\, dx$$

For simplicity, we put

$$\Psi(t,x) := t_0 + t + \langle x\rangle^{2-\alpha}.$$

Lemma 6

$$\begin{aligned}\frac{d}{dt}E_1(t) = &-\int_\Omega a(x)|u_t|^2\Psi^{\lambda+\frac{\alpha}{2-\alpha}}\, dx\\ &-\left(\lambda + \frac{\alpha}{2-\alpha}\right)\int_\Omega u_t(\nabla_x u \cdot \nabla_x \Psi)\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\, dx\\ &+\frac{1}{2}\left(\lambda + \frac{\alpha}{2-\alpha}\right)\int_\Omega (|u_t|^2 + |\nabla_x u|^2)\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\, dx.\end{aligned}$$

$$E_0(t) := \int_\Omega \left(u u_t + \frac{a(x)}{2}|u|^2 \right) \Phi(t,x)^{-1+2\delta}\, dx$$

Lemma 7

$$\begin{aligned}
\frac{d}{dt}E_0(t) \le &-\frac{\delta}{1-\delta}\int_\Omega |\nabla_x u|^2 \Phi^{-1+2\delta}\, dx \\
&- \frac{1-2\delta}{2}\int_\Omega |u|^2 \Phi^{-1+2\delta}\cdot c\tilde{a}(x)\Psi^{-\beta-1}\, dx \\
&+ \int_\Omega |u_t|^2 \Phi^{-1+2\delta}\, dx \\
&- (1-2\delta)\int_\Omega u u_t \Phi^{-2+2\delta}\partial_t \Phi\, dx \\
&- \frac{1-2\delta}{2}\int_\Omega |u|^2 \Phi^{-1+2\delta}(a(x)-\tilde{a}(x))\partial_t \Phi\, dx.
\end{aligned}$$

Lemmas 6 & 7 are essentially given in Sobajima–W. (2021).

The crucial point is to apply Lemma 4 with

$$h(t,x) := \begin{cases} \mu\Phi(t,x)^{-1+2\delta} x & (|x| < L), \\ \mu\Phi(t,x)^{-1+2\delta} \dfrac{L}{|x|} x & (|x| \geq L), \end{cases}$$

where $\mu > 0$ is a suitably small constant.
Then, we have:

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u_t h(t,x) \cdot \nabla_x u \, dx \\ &= -\frac{1}{2} \int_\Omega (\nabla_x \cdot h) \left(|u_t|^2 - |\nabla_x u|^2 \right) dx + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \nu \cdot h \, dS \\ &\quad - \int_\Omega \sum_{i,j=1}^n \partial_i u \partial_j u \partial_i h_j \, dx - \int_\Omega a(x) u_t (h \cdot \nabla_x u) \, dx \\ &\quad - \int_\Omega u \partial_t h(t,x) \cdot \nabla_x u \, dx. \end{aligned}$$

Finally, we define the modified energy by

$$\mathcal{E}(t) := \int_\Omega u_t h(t,x) \cdot \nabla_x u \, dx + \nu E_0(t) + E_1(t),$$

where $\nu > 0$ is a suitably small constant.

Then, we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) \le & \int_\Omega \left(-\frac{1}{2}\nabla_x \cdot h(t,x) + \nu \Phi^{-1+2\delta} - a(x)\Psi^{\lambda + \frac{\alpha}{2-\alpha}} \right) |u_t|^2 \, dx \\ & + \int_\Omega \left(\frac{1}{2}\nabla_x h(t,x) - \nu \frac{\delta}{1-\delta}\Phi^{-1+2\delta} \right) |\nabla_x u|^2 \, dx \\ & + \cdots \end{aligned}$$

Taking the parameters $\delta, \mu, \nu, \beta, t_0$ appropriately (in this step, we use the assumption $\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1}))$, we can get (RHS) ≤ 0. □

Summary

(4) $$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \Omega. \end{cases}$$

$a \in C^\infty(\overline{\Omega})$, $a(x) \geq 0$, and there exist $\alpha \in [0,1)$ and $a_0 > 0$ such that

$$\lim_{|x|\to\infty} |x|^\alpha a(x) = a_0.$$

Theorem 5

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies

$$I_0 := \int_\Omega \left(|u_1(x)|^2 + |\nabla_x u_0(x)|^2 + |u_0(x)|^2 \langle x \rangle^{-\alpha} \right) \langle x \rangle^{\lambda(2-\alpha)}\, dx < +\infty$$

with some $\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1})$, then the solution u to (4) satisfies

$$E(t) \leq C(1+t)^{-\lambda-1} I_0 \qquad (t \geq 0).$$