Himeji Conference on Partial Differential Equations
Titles and abstracts 2021

FURUTANI, Kenro (Advanced Mathematical Institute, Osaka City University)

Title: **Calabi-Yau structure and a Bargmann type transformation on the Cayley projective plane**

Abstract: The purpose of our talk is

(1) to show the existence of a *Calabi-Yau structure* on the punctured cotangent bundle $T_0^*(P^2\mathbb{O})$ of the Cayley projective plane $P^2\mathbb{O}$,

(2) to construct a *Bargmann type transformation* between a space of holomorphic functions on $T_0^*(P^2\mathbb{O})$ and the L_2-space on $P^2\mathbb{O}$.

A Kähler structure on $T_0^*(P^2\mathbb{O})$ was shown by identifying it with a quadrics in the complex space $\mathbb{C}^{27}\backslash\{0\}$ and the natural symplectic form of the cotangent bundle $T_0^*(P^2\mathbb{O})$ is expressed as a Kähler form.

Our method to construct the transformation is the pairing of polarizations, one is the natural Lagrangian foliation given by the projection map $\mathbf{q} : T_0^*(P^2\mathbb{O}) \longrightarrow P^2\mathbb{O}$ and the positive complex polarization defined by the Kähler structure.

The transformation gives a quantization of the geodesic flow in terms of one parameter group of elliptic Fourier integral operators whose canonical relations are defined by the graphs of the geodesic flow action at each time. It turn out that for the Cayley projective plane the results are a little bit different from other cases of the original Bargmann transformation for Euclidean space, spheres and other projective spaces.

This is based on a joint work with Kurando Baba (TUS): arXiv 2101.07505: Calabi-Yau structure and Bargmann type transformation on the Cayley projective plane.

HIGUCHI, Kenta (Ritsumeikan University)

Title: **Semiclassical resonances generated by crossings of classical trajectories**

Abstract: We consider a one-dimensional semiclassical coupled Schrödinger operator

$$P(h) = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix} \qquad \text{in} \quad L^2(\mathbb{R}) \oplus L^2(\mathbb{R}). \tag{1}$$

Here $h > 0$ denotes the semiclassical parameter,

$$P_j = -h^2 \frac{d^2}{dx^2} + V_j(x), \quad x \in \mathbb{R}, \ (j = 1, 2)$$

are Schrödinger operators with $V_j \in C^\infty(\mathbb{R};\mathbb{R})$, $W = W(x, hD_x)$ a first-order semiclassical differential operator and W^* its formal adjoint. We study the resonances of P near an energy $E_0 \in \mathbb{R}$ at which both the Hamiltonian systems corresponding to P_1 and P_2 are non-trapping. It is well known that, near E_0, each P_j has no resonance with imaginary part of order $h \log(1/h)$ (see [Ma]). We will see however that the coupled operator $P(h)$

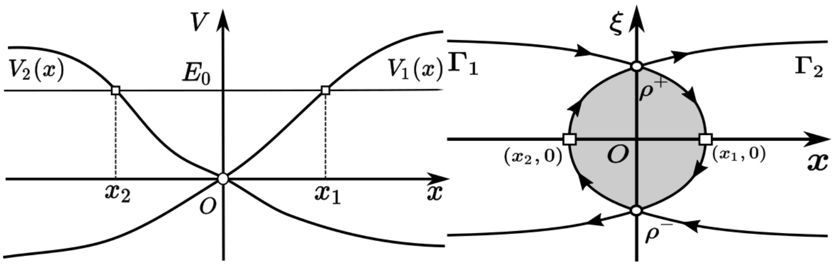

Figure 1: A case with a closed curve

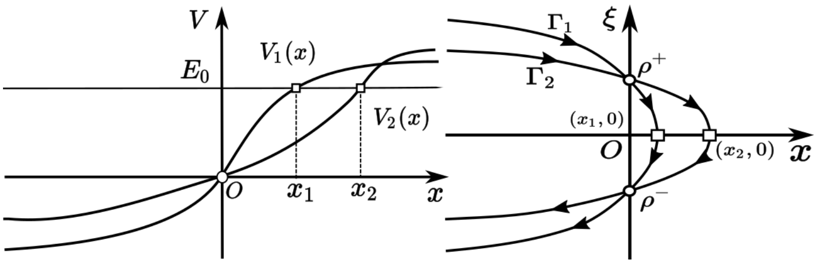

Figure 2: A case without any closed curves

has resonances at this order if the Hamiltonian trajectories corresponding to P_1 and P_2 cross and make a closed trajectory. We give precise asymptotics of such resonances in terms of the geometry of these trajectories and the crossings.

[Hi] K. Higuchi, *Resonance free domain for a system of Schrödinger operators with energy-level crossings*, Rev. Math. Phys., Online Ready.
[Ma] A. Martinez, *Resonance free domains for non globally analytic potentials*, Annales Henri Poincaré **3** (2002), 739–756.

HIROSHIMA, Fumio (Kyushu University)

Title: **Localization of the ground state of the Nelson model**

Abstract: The ground state of the renormalized and non-renormalized Nelson model in quantum field theory is studied. Localizations of the ground state are an important problem. The ground state can be seen as a function of three argumetns: the position variable x, the number of bosons n and the field variable ϕ. We show the spatial exponential decay in x in terms of the Agmon metric and Gaussian domination in ϕ. The super exponential decay of the number of bosons is also shown. To show the localizations we apply the Gibbs measure associated with the ground state.

ISHIWATA, Satoshi (Yamagata University)

Title: **Geometric analysis on manifolds with ends**

Abstract: In this talk, we consider the behavior of the heat kernel on manifold with ends (ex. connected sum of tube and plane) and some related results. This talk is based on a joint work with A. Grigor'yan (Bielefeld) and L. Saloff-Coste (Cornell).

KAMEOKA, Kentaro (The University of Tokyo)

Title: **Resonances and complex absorbing potential method for the Wigner-von Neumann type Hamiltonian**

Abstract: We study the Wigner-von Neumann type Hamiltonian, which has an oscillatory and slowly decaying potential. We define the complex resonances by proving the existence of meromorphic continuations of matrix elements of resolvent. We also characterize resonances by complex absorbing potential method. The proofs are based on our new complex distortion of Hamiltonian. This is joint work with Shu Nakamura.

KAMIMOTO, Joe (Kyushu University)

Title: **Asymptotic analysis of oscillatory integrals with degenerate phases**

Abstract: This talk is concerned with the behavior of scalar oscillatory integrals of the form:

$$I(t) = \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx$$

for large values of the real papameter t. The above integrals appear in various mathematical fields and there have been many interesting studies about their behavior.

In the case when the Hessian of the phase is nondegenerate, the phase can be simply expressed by coordinate changes (via the Morse lemma), which can compute an exact asymptotic expansion of oscillatory integrals. On the other hand, when the phase is degenerate, it is quite difficult to understand their bahavior since regular coordinate changes are not always sufficiently useful in this case. In the history of studies in the degenerate case, methods based on "resolution of singularities" provide extremely powerful tools for understanding their behavior. The seminal result in this direction is due to A. Varchenko. In this talk, after recalling his work, we explain many kinds of results which generalize and improve it. These contain recent important works due to the real analysis school of E.M. Stein and our results with T. Nose.

LEVITINA, Galina (Australian National University)

Title: **Cwikel-type estimates on open domains**

Abstract: Estimates on the rate of decay of the singular values of the operators of the type f(x)g(-iv) take their origin in the study of the number of bound states of Schrodinger operators. Estimates in the weak-Schatten ideals $\mathcal{L}_{p,\infty}$, were first conjectured by Simon and later proved by Cwikel for $p > 2$. In this talk Cwikel-type estimates in weak-Schatten ideal $\mathcal{L}_{1,\infty}$ in bounded open sets of the Euclidean space are discussed. Different boundary conditions for a self-adjoint Laplacian are considered.

NAKANO, Fumihiko (Tohoku University)

Title: **Scaling limit of eigenvalues and eigenfunctions of 1-dimensional random Schrödinger operators**

Abstract: 1-dimensional Schrödinger operator with random decaying potential has various spectral and level statistics properties depending on the decay rate of the potential at infinity. In this talk we consider the random measure associated to the eigenfunctions and its scaling limit.

NIIKUNI, Hiroaki (Maebashi Institute of Technology)

Title: **Edge states of Schrödinger equations on graphene with zigzag boundaries**

Abstract: Recently, topological insulators have been garnering attention in condensed matter physics. They behave as an insulator in its interior, but their surfaces contain conducting states (edge states). These properties are explained by comparing the spectum of a Hamiltonian in the whole space with it in the half space with a boundary.

In this talk, we discuss spectral structures of Schrödinger operators on graphene with zigzag boundaries from the point of view of quantum graphs. Let $\Gamma = (E, V)$ be the hexagonal lattice (graphene), $\Gamma^\sharp = (E^\sharp, V^\sharp)$ the half of Γ with zigzag boundaries. Here, E ($E^\sharp$, resp.) and V ($V^\sharp$, resp.) are the set of edges and vertices of Γ ($\Gamma^\sharp$, resp.), respectively. Assume that the length of each edge is equal to 1. We define the Hamiltonian H in $L^2(\Gamma)$ and $H^\sharp$ in $L^2(\Gamma^\sharp)$ as follows. For each $e \in E^\sharp$,

$$(H^\sharp y)(x) = -y''(x) + q(x)y(x),$$

where $x \in (0, 1)$ is the local coordinate on e and $q \in L^2(0, 1)$ is a potential. Let $y \in \mathrm{Dom}(H^\sharp)$ satisfy (i) the Kirchhoff–Neumann vertex condition at any $v \in V^\sharp$ except zigzag boundaries and (ii) the Dirichlet boundary condition on zigzag boundaries. On the other hand, the operator H acts as

$$(Hy)(x) = -y''(x) + q(x)y(x)$$

on each $x \in E$. In addition, $y \in \mathrm{Dom}(H)$ satisfies the Kirchhoff–Neumann vertex condition at any $v \in V$. If q is even, then the spectrum of H is discussed by P. Kuchment and O. Post in 2007. Thus, we study $\sigma(H^\sharp)$ for a general $q \in L^2(0, 1)$. Then, we compare $\sigma(H^\sharp)$ with $\sigma(H)$ for an even $q \in L^2(0, 1)$.

Using the theory of direct integral decompositions, we have a unitary equivalence $H^\sharp \simeq \int_{S^1}^{\oplus} H^\sharp(\mu) \frac{d\mu}{2\pi}$, where $H^\sharp(\mu)$ is a fiber operator for $H^\sharp$ for a quasi-momentum $\mu \in S^1 := [-\pi, \pi)$. Studying $\sigma(H^\sharp(\mu))$ for each $\mu \in S^1$, we derive the results on $\sigma(H^\sharp)$. To state main results, we introduce notations. Let σ_D be the set of eigenvalues of the problem $-y'' + qy = \lambda y$ and $y(0) = y(1)$. Moreover, we put $\sigma_p^\sharp = \sigma_p(H^\sharp) \setminus \sigma_D$ and call an eigenfunction corresponding to $\lambda \in \sigma_p^\sharp$ an edge state.

Theorem 1 *(to appear in "Results in Mathematics") Fix $q \in L^2(0,1)$, which is not necessarily even.*

(i) (Basic spectral structure) There exists some sequence $\lambda_0^+ < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \cdots < \lambda_j^- \leq \lambda_j^+ < \cdots \to +\infty$ such that

$$\sigma(H^\sharp) = (\bigcup_{j=1}^{\infty} B_j) \cup \sigma_D \cup \sigma_p^\sharp,$$

where $B_j = [\lambda_{j-1}^+, \lambda_j^-]$ for each $j \in \mathbb{N}$.

(ii) (Existence of edge states) We have $\sigma_p^\sharp = \{\lambda \in \mathbb{R} | \quad \theta(1,\lambda) + 2\varphi'(1,\lambda) = 0\}$ as the infinite set, where $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ are the solutions to $-y'' + qy = \lambda y$ as well as the initial conditions $(\theta(0,\lambda), \theta'(0,\lambda)) = (1,0)$ and $(\varphi(0,\lambda), \varphi'(0,\lambda)) = (0,1)$, respectively. In particular, we have $\sigma_p^\sharp \neq \emptyset$.

(iii) (Location of the eigenvalues) Putting $G_j = (\lambda_j^-, \lambda_j^+)$, $\partial G_j = \{\lambda_j^-, \lambda_j^+\}$ and $\overline{G_j} = [\lambda_j^-, \lambda_j^+]$ for each $j \in \mathbb{N}$, we have

$$\sigma_D \subset \bigcup_{n=1}^{\infty} \overline{G_{2n}} \quad \text{and} \quad \sigma_p^\sharp \subset \bigcup_{n=1}^{\infty} \overline{G_{2n-1}}.$$

Remark 1 *In 2007, P. Kuchment and O. Post proved that $\sigma(H) = (\bigcup_{j=1}^{\infty} B_j) \cup \sigma_D$, provided that $q \in L^2(0,1)$ is even. In this sense, $\sigma_p^\sharp$ stands the set of the eigenvalues for the edge states.*

SHIMAKURA, Norio

Title: **Resolvent kernel of an elliptic operator**

Abstract: – See Appendix –

TAKEI, Yumiko (Kwansei Gakuin University)

Title: **WKB analysis via topological recursion for hypergeometric differential equations**

Abstract: The exact WKB analysis is a method to analyze ordinary differential equations with a small parameter $\hbar$. The main ingredient of the exact WKB analysis is a formal solution for $\hbar$, called a WKB solution. When we study differential equations by using the exact WKB analysis, the so-called Voros coefficients play an important role. The Voros coefficient is defined as a contour integral of the logarithmic derivative of WKB solutions.

On the other hand, the topological recursion is introduced by B. Eynard and N. Orantin [EO] to study the correlation functions in the random matrix theory and it gives a generalization of the loop equations for the matrix model.

Recently, a surprising connection between the WKB analysis and the topological recursion has been discovered, that is, it is found that WKB solutions can be constructed via the topological recursion [BE].

In this talk, we prove that the Voros coefficients for hypergeometric differential equations are described by the generating functions of free energies defined in terms of the topological recursion. Furthermore, as its applications we show the following objects can be explicitly computed for hypergeometric equations: (i) three-term difference equations that the generating function of free energies satisfies, (ii) explicit form of the free energies, and (iii) explicit form of Voros coefficients [IKT,T].

[BE] Bouchard, V. and Eyanard, B., *Reconstructing WKB from topological recursion*, Journal de l'Ecole polytechnique – Mathematiques, **4** (2017), 845–908.
[EO] Eynard, B. and Orantin, N., *Invariants of algebraic curves and topological expansion*, Communications in Number Theory and Physics, **1** (2007), 347–452.
[IKT] Iwaki, K., Koike, T., and Takei, Y.-M., *Voros coefficients for the hypergeometric differential equations and Eynard-Orantin's topological recursion*, Part I, arXiv:1805.10945 & Part II, Journal of Integrable Systems **3** (2019), 1–46.
[T] Takei, Y.-M., *Voros Coefficients and the Topological Recursion for a Class of the Hypergeometric Differential Equations associated with the Degeneration of the 2- dimensional Garnier System*, arXiv: 2005.08957.

Organized by : FUJIIE, S., HOSHIRO, T., ISOZAKI, H., IWASAKI, C., NAGAYASU, S., NOMURA, Y., RICHARD, S., UMEDA, T., and WATANABE, T.

Resolvent Kernel of an Elliptic Operator

Norio SHIMAKURA　島倉紀夫

(ある楕円型作用素のレゾルヴェント核, 偏微分方程式姫路研究集会, 2021 年 3 月 4-6 日 (水-金))

Let $x=(x_1,\cdots,x_n)$ be a vector of $n(\geq 1)$ real variables and $\{h_p\}_{p=0}^n$ be real numbers independent of x such that

$$h_p \geq 1/2 \quad (0\leq p\leq n) \quad \text{and that} \quad \textstyle\sum_{p=0}^n h_p > (n+1)/2. \tag{1}$$

We fix the notation of n, x and $\{h_p\}_{p=0}^n$ once for all.

Let $\mathcal{L}$ be a partial differential operator of variables x:

$$\mathcal{L}u = -\sum_{j,k=1}^{n}(\delta_{jk}x_j - x_j x_k)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n}\left\{\left(\sum_{p=0}^{n} h_p\right)x_j - h_j\right\}\frac{\partial u}{\partial x_j} \tag{2}$$

which transforms a complex function $u(x)$ to another one [3]. We set

$$\kappa = -1 + \sum_{p=0}^{n} h_p \ (>0).$$

$\mathcal{L}$ is symmetric in the simplex

$$\Omega = \{x\in \mathbf{R}^n; \quad x_j>0 \ (1\leq j\leq n) \ \text{ and } \ x_0 = 1 - \textstyle\sum_{j=1}^n x_j > 0\} \tag{3}$$

with respect to the scalar product

$$(u,v) = \int_\Omega u(x)\overline{v(x)}\,dV(x) \quad \text{with} \quad dV(x) = \left(\prod_{p=0}^{n} {x_p}^{h_p-1}\right)dx_1\cdots dx_n \tag{4}$$

on the complex Hilbert space $L^2(\Omega, dV)$. The symbol of $\mathcal{L}$ is equal to

$$\Delta(\xi,x) = \sum_{j,k=1}^{n}(\delta_{jk}x_j - x_j x_k)\xi_j\xi_k \quad \text{and} \quad \det\Delta(\xi,x) = \prod_{p=0}^{n} x_p.$$

$\Delta(\xi,x)$ is positive definite in Ω and is of rank $n-1$ on the boundary of Ω. Define a second scaler product

$$((u,v)) = \int_\Omega\left\{u\,\overline{v} + \sum_{j,k=1}^{n}(\delta_{jk}x_j - x_j x_k)\frac{\partial u}{\partial x_j}\frac{\partial \overline{v}}{\partial x_k} + \mathcal{L}u\,\overline{\mathcal{L}v}\right\}dV(x) \tag{4'}$$

for polynomials u, v. The completion $D(\mathbf{L})$ of the set of polynomials endowed with the scalar product $(\!(\ , \ )\!)$ is a dense linear subspace of $L^2(\Omega, dV)$. And $D(\mathbf{L})$ is the set of $u \in L^2(\Omega, dV)$ for which $(\!(u, u)\!) < +\infty$. Set $\mathbf{L}u = \mathcal{L}u$ for $u \in D(\mathbf{L})$. $\mathbf{L}$ is said to be the **Friedrichs extension** of $\mathcal{L}$.

Let $\mathcal{N} = \{0, 1, 2, \cdots\}$ be the set of non-negative integers and

$$l_s = s^2 + \kappa s \quad \text{for} \quad s \in \mathcal{N}. \tag{5}$$

Given an $s \in \mathcal{N}$ and a monomial $u_1(x)$ of degree s, there exists a unique polynomial $u_2(x)$ of degree less than s such that $u = u_1 + u_2$ satisfies the equation $\mathbf{L}u = l_s u$. The linear subspace $\mathcal{E}_s$ of $D(\mathbf{L})$ spanned by polynomials u satisfying $\mathbf{L}u = l_s u$ is of dimension $(n-1+s)!/\{(n-1)!s!\}$, number of monomials of degree s in n variables.

If $\lambda \notin \{-\lambda_s\}_{s=0}^{\infty}$ and if $f \in L^2(\Omega, dV)$, there exists a unique $u \in D(\mathbf{L})$ which satisfies $\mathbf{L}u + \lambda u = f$. The mapping of f to u is linear and continuous from $L^2(\Omega, dV)$ onto $\mathbf{D(L)}$. Denote $u = G_\lambda f = (\mathbf{L} + \lambda)^{-1} f$. $G_\lambda = (\mathbf{L} + \lambda)^{-1}$ is said to be the **resolvent** of $\mathbf{L} + \lambda$. This is by definition

$$G_\lambda = (\mathbf{L} + \lambda)^{-1} = \sum_{s=0}^{\infty} \frac{1}{l_s + \lambda} \mathbf{E}_s \quad \text{for} \quad \lambda \notin \{-l_s\}_{s=0}^{\infty}, \tag{6}$$

where $\mathbf{E}_s$ is the orthogonal projection of $L^2(\Omega, dV)$ onto $\mathcal{E}_s$. G_λ admits a unique kernel representation $G(x, y; \lambda)$ and we have

$$(\mathbf{L} + \lambda)^{-1} f(x) = \int_\Omega G(x, y; \lambda) f(y) dV(y). \tag{7}$$

Theorem. The kernel of the resolvent $\{\mathbf{L} + \frac{1}{4}(\kappa^2 + \xi^2)\}^{-1}$ is

$$G\Big(x, y; \frac{\kappa^2 + \xi^2}{4}\Big) = \int_0^\pi \cdots \int_0^\pi {}_2F_1\Big(\frac{\kappa + i\xi}{2}, \frac{\kappa - i\xi}{2}; \frac{1}{2}; \Big(\sum_{p=0}^{n} \sqrt{x_p y_p} \cos \phi_p\Big)^2\Big)$$

$$\times \frac{1}{16} \Gamma\Big(\frac{\kappa + i\xi}{2}\Big) \Gamma\Big(\frac{\kappa - i\xi}{2}\Big) \prod_{q=0}^{n} \frac{(\sin \phi_q)^{2h_q - 2} d\phi_q}{\sqrt{\pi}\, \Gamma(h_q - \frac{1}{2})}, \tag{8}$$

where ${}_2F_1(a,b;c;\zeta)$ is the hypergeometric series of Gauss

$${}_2F_1(a,b;c;\zeta)=\sum_{s=0}^{\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+s)s!}\zeta^s.$$

The simplest is in the case where $\sum_{p=0}^{n} h_p=2$:

$$G\Big(x,y;\frac{1+\xi^2}{4}\Big)=\int_0^{\pi}\cdots\int_0^{\pi}\frac{\cos(\xi\sum_{p=0}^{n}\sqrt{x_py_p}\cos\phi_p)}{\cos(\sum_{p=0}^{n}\sqrt{x_py_p}\cos\phi_p)}\prod_{q=0}^{n}\frac{(\sin\phi_q)^{2h_q-2}d\phi_q}{\sqrt{\pi}\,\Gamma(h_q-\frac{1}{2})} \qquad (9)$$

obtained by applying a formula

$${}_2F_1\Big(\frac{1+\xi}{2},\frac{1-\xi}{2};\frac{1}{2};(\sin z)^2\Big)=\frac{\cos(\xi z)}{\cos z}$$

to $\kappa=1$ [1,p.101]. A fundamental is a generalization of the β-function

$$\int_{\Omega}\prod_{p=0}^{n}{x_p}^{\alpha_p}dV(x)=\frac{\prod_{p=0}^{n}\Gamma(\alpha_p+h_p)}{\Gamma(\sum_{p=0}^{n}(\alpha_p+h_p))}\quad \text{for}\quad (\alpha_0,\alpha_1,\cdots,\alpha_n)\in\mathcal{N}^{n+1}.$$

We begin by the kernel $E_s(x,y)$ of $\mathbf{E}_s$ $(s\in\mathcal{N})$. It is a symmetric polynomial of x,y of degree s with respect to each of x and y [3,p.294,(4.11)],

$$\frac{E_s(x,y)}{2s+\kappa}=\Gamma(\kappa)\int_0^{\pi}\cdots\int_0^{\pi}C_{2s}^{(\kappa)}\Big(\sum_{p=0}^{n}\sqrt{x_py_p}\cos\phi_p\Big)dM(\phi), \qquad (10)$$

where $dM(\phi)$ is the measure on $(0,\pi)^{n+1}$ defined to be

$$dM(\phi)=\prod_{q=0}^{n}\frac{(\sin\phi_q)^{2h_q-2}d\phi_q}{\sqrt{\pi}\,\Gamma(h_q-\frac{1}{2})}\quad [3,(4.8)] \qquad (11)$$

and $C_{2s}^{(\kappa)}(z)$ are the Gegenbauer polynomials of even degree generated by a convergent power series

$$\sum_{s=0}^{\infty}C_{2s}^{(\kappa)}(z)t^{2s}=\frac{1}{2}\{(1-2zt+t^2)^{-\kappa}+(1+2zt+t^2)^{-\kappa}\} \qquad (12)$$

for $-1 \leq z \leq 1$ and $|t|<1$. Set

$$\mathcal{F}(\theta, z)=(1/2)\Gamma(\kappa)\{(\cosh\theta-z)^{-\kappa}+(\cosh\theta+z)^{-\kappa}\}. \tag{13}$$

From (9) and (12), we have a generating function of $\{E_s(x,y)\}_{s=0}^{\infty}$:

$$\sum_{s=0}^{\infty} e^{-(2s+\kappa)|\theta|}\frac{E_s(x,y)}{2s+\kappa}=2^{-\kappa}\int_0^{\pi}\cdots\int_0^{\pi}\mathcal{F}\Big(\theta,\sum_{p=0}^{n}\sqrt{x_p y_p}\cos\phi_p\Big)dM(\phi), \tag{14}$$

where θ is a complex variable independent of x. (12) and (14) imply

$$\mathcal{F}(\theta, z)=\Gamma(\kappa)\sum_{s=0}^{\infty} e^{-(2s+\kappa)|\theta|}C_{2s}^{(\kappa)}(z). \tag{15}$$

Both sides of (14) converge on the set $\mathbf{R}_\theta \times \{-1\leq z\leq 1\}$ without probably on the subset $\{(\theta, z);\ \ \theta=0 \text{ and } z=\pm 1\}$ [3,(4.8)].

Let us establish the kernel $G(x, y; \lambda)$ in the case where

$$\lambda=(\kappa^2+\xi^2)/4 \quad \text{avec} \quad \xi\in\mathbf{R}. \tag{16}$$

We need again a sequence $\{k_s(\xi)\}_{s=0}^{\infty}$:

$$k_s(\xi)=\frac{1}{2}\int_{-\infty}^{\infty} e^{-(2s+\kappa)|\theta|+i\xi\theta}d\theta=\frac{2s+\kappa}{4l_s+\kappa^2+\xi^2} \quad (s\in\mathcal{N}). \tag{17}$$

Multiply $e^{i\xi\theta}$ to both sides of (15) and integrate with respect to θ. Then,

$$G\Big(x, y; \frac{\kappa^2+\xi^2}{4}\Big)=\sum_{s=0}^{\infty}\frac{4E_s(x,y)}{4l_s+\kappa^2+\xi^2}=\int_0^{\pi}\cdots\int_0^{\pi}2^{-\kappa}\mathcal{H}\Big(\xi,\sum_{p=0}^{n}\sqrt{x_p y_p}\cos\phi_p\Big)dM(\phi), \tag{18}$$

where $dM(\phi)$ is defined by (11) and

$$\mathcal{H}(\xi, z)=\int_{-\infty}^{+\infty}\mathcal{F}(\theta, z)e^{i\xi\theta}d\theta \quad \text{for} \quad (\xi, z)\in\mathbf{R}\times\{-1<z<1\}. \tag{19}$$

$\mathcal{F}(\theta, z)$ and $\mathcal{H}(\xi, z)$ are rapidly decreasing even functions of θ and of ξ, respectively.

We define a Jacobi type polynomial by the O.Rodrigues formula

$$P_\alpha(x)=\Big(\prod_{p=0}^{n}x_p^{1-h_p}\Big)\frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\Big\{\Big(\prod_{q=0}^{n}x_q^{h_q-1}\Big)x_0^{\alpha_1+\cdots+\alpha_n}x_1^{\alpha_1}\cdots x_n^{\alpha_n}\Big\}. \quad (20)$$

It is represented by a multiple complex integral

$$P_\alpha(x)=\frac{\alpha_1!\cdots\alpha_n!}{(2\pi i)^n x_0^{h_0-1}}\int\cdots\int\Big(1-\sum_{j=1}^{n}\zeta_j\Big)^{\alpha_1+\cdots+\alpha_n+h_0-1}\prod_{k=1}^{n}\frac{\zeta_k^{\alpha_k+h_k-1}d\zeta_k}{(\zeta_k-x_k)^{\alpha_k+1}}.$$

We set $\eta_j=\zeta_j/x_j$ and $\gamma_j=\{|\eta_j-1|=\varepsilon\}$ to have

$$P_\alpha(x)=\frac{\alpha_1!\cdots\alpha_n!}{(2\pi i)^n x_0^{h_0-1}}\int_{\gamma_1}\cdots\int_{\gamma_n}\Big(1-\sum_{j=1}^{n}x_j\eta_j\Big)^{\alpha_1+\cdots+\alpha_n+h_0-1}\prod_{k=1}^{n}\frac{\eta_k^{\alpha_k+h_k-1}d\eta_k}{(\eta_k-1)^{\alpha_k+1}}.$$

Erase the sign of integration and we have finally

$$P_\alpha(x)=x_0^{1-h_0}\Big(\prod_{i=1}^{n}\alpha_i!^2\frac{\partial^{\alpha_i}}{\partial\eta_i^{\alpha_i}}\Big)\Big[\Big(1-\sum_{j=1}^{n}x_j\eta_j\Big)^{\alpha_1+\cdots+\alpha_n+h_0-1}\prod_{k=1}^{n}\eta_k^{\alpha_k+h_k-1}\Big]\Big|_{\eta=(1,\cdots,1)} \quad (21)$$

Successive differentiations are done with respect to x in (20), while they are done with respect to η in (21) at $\eta=(1,\cdots,1)$. $P_\alpha(x)$ is of degree

$$|\alpha|=\alpha_1+\cdots+\alpha_n$$

and satisfies the equation $\mathcal{L}[P_\alpha]=l_{|\alpha|}P_\alpha$ [6,p.306]. Any polynomial of $(x_1,\cdots,x_n)$ is equal to one and only one linear combination of $P_\alpha(x)$'s.

On the other hand, let us define a second polynomial

$$Q_\alpha(x)=C_\alpha\Big[x^\alpha+\sum_{k=1}^{|\alpha|}\Big(\prod_{r=1}^{k}\frac{1}{l_{|\alpha|-r}-l_{|\alpha|}}\Big)\Big\{\sum_{j=1}^{n}\Big(x_j\frac{\partial^2}{\partial x_j^2}+h_j\frac{\partial}{\partial x_j}\Big)\Big\}^k x^\alpha\Big] \quad (22)$$

for every $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathcal{N}^n$, where the coefficient C_α is the following

$$(-1)^{|\alpha|}\Big/\Big(C_\alpha\prod_{k=1}^{n}\alpha_k!\Big)=\int\cdots\int_{y_0>0,y_1>0,\cdots,y_n>0,y_0+y_1+\cdots+y_n<1}y_0^{h_0-1+\alpha_1+\cdots+\alpha_n}y_1^{h_1-1+\alpha_1}\cdots y_n^{h_n-1+\alpha_n}dy_0\cdots dy_n. \quad (23)$$

$Q_\alpha(x)$ is also of degree $|\alpha|$ and satisfies $\mathcal{L}[Q_\alpha] = l_{|\alpha|}Q_\alpha$ [3,(B.2),p.306]. $Q_\alpha(x)$ contains an only one term x^α of degree $|\alpha|$ and any polynomial of x is equal to a linear combination of $Q_\alpha(x)$'s.

We have by (15) and (19)

$$\mathcal{H}(\xi,z)=\frac{\Gamma(\kappa)}{2^{1+\kappa}}\int_0^1(t^{i\xi}+t^{-i\xi})\sum_{s=0}^{\infty}C_{2s}^{(\kappa)}(z)t^{2s+\kappa-1}dt=\frac{\Gamma(\kappa)}{2^\kappa}\sum_{s=0}^{\infty}\frac{2s+\kappa}{4l_s+\kappa^2+\xi^2}C_{2s}^{(\kappa)}(z)$$

which is the Laurent expansion of $\mathcal{H}(\xi,z)$ or again

$$\mathcal{H}\Big(\xi,\sum_{p=0}^{n}\sqrt{x_py_p}\cos\phi_p\Big)=\frac{\Gamma(\kappa)}{2^\kappa}\sum_{s=0}^{\infty}\frac{2s+\kappa}{4l_s+\kappa^2+\xi^2}C_{2s}^{(\kappa)}\Big(\sum_{p=0}^{n}\sqrt{x_py_p}\cos\phi_p\Big). \quad (24)$$

Lemma 1 For every non-gegative integer s, the integral

$$\int_0^\pi\cdots\int_0^\pi C_{2s}^{(\kappa)}\Big(\sum_{p=0}^{n}\sqrt{x_py_p}\cos\phi_p\Big)dM(\phi) \quad (a)$$

is a symmetric polynomial of x,y.

Proof. (a) is a linear combination of integrals

$$\int_0^\pi\cdots\int_0^\pi\prod_{p=0}^{n}\{(\sqrt{x_py_p}\cos\phi_p)^{m_p}(\sin\phi_p)^{2h_p-2}\}dM(\phi). \quad (b)$$

If one of them does not vanish identically, $\sum_{p=0}^{n}m_p$ is even because $C_{2s}^{(\kappa)}(\zeta)$ is an even polynomial. If one of m_p is odd, the corresponding integral (b) vanishes identically as a function of x,y because

$$\int_0^\pi(\sqrt{x_py_p}\cos\phi_p)^{m_p}(\sin\phi_p)^{2h_p-2}d\phi_p=0. \quad (c)$$

So, we have only the terms of type (b) with all m_p even and all square root signs dispparear by integration on $(0,\pi)^{n+1}$. The function (a) is therefore a symmetric polynomial of x,y. **q.e.d.**

Lemma 2

$$\mathcal{H}(\xi,z)=2^{\kappa-4}\Gamma\Big(\frac{\kappa+i\xi}{2}\Big)\Gamma\Big(\frac{\kappa-i\xi}{2}\Big)\,{}_2F_1\Big(\frac{\kappa+i\xi}{2},\frac{\kappa-i\xi}{2};\frac{1}{2};z^2\Big). \quad (25)$$

Proof. Define a new variable ω instead of θ by setting

$$\sin\omega = 1/\cosh\theta \quad \text{with} \quad \omega|_{\theta=0} = \pi/2.$$

Then, $d\theta = (1/\sin\omega)d\omega$ and we have by (11)

$$\mathcal{H}(\xi, z) = \frac{1}{2}\Gamma(\kappa)\int_0^\pi \left(\tan\frac{\omega}{2}\right)^{i\xi}\left\{\frac{(\sin\omega)^{\kappa-1}}{(1-z\sin\omega)^\kappa} + \frac{(\sin\omega)^{\kappa-1}}{(1+z\sin\omega)^\kappa}\right\}d\omega.$$

It is an even function of z whose Taylor series is

$$\mathcal{H}(\xi, z) = \sum_{s=0}^{\infty}\frac{\Gamma(\kappa+2s)}{(2s)!}z^{2s}\int_0^\pi \left(\tan\frac{\omega}{2}\right)^{i\xi}(\sin\omega)^{\kappa+2s-1}d\omega.$$

Set $u = (\tan\frac{\omega}{2})^2$. Then,

$$\int_0^\pi \left(\tan\frac{\omega}{2}\right)^{i\xi}(\sin\omega)^{\kappa+2s-1}d\omega = 2^{\kappa+2s-4}\int_0^{+\infty} u^{\frac{i\xi+\kappa}{2}+s-1}(1+u)^{-\kappa-2s}du$$

$$= 2^{\kappa+2s-4}\Gamma\left(\frac{\kappa+i\xi}{2}+s\right)\Gamma\left(\frac{\kappa-i\xi}{2}+s\right)\Big/\Gamma(\kappa+2s).$$

The expression (25) of $\mathcal{H}(\xi, z)$ is obtained. **q.e.d.**

Proof of the Theorem. (8) follows at once from (18),(25). **q.e.d.**

Let us verify the resolvent equation :

$$(\lambda-\mu)\int_\Omega G(x, x'; \mu)G(x', y; \lambda)dV(x') = G(x, y; \mu) - G(x, y; \lambda). \tag{26}$$

Proof of (26). Given two arbitrary real numbers ξ, η, set for simplicity

$$G_\lambda(x, y) = G(x, y; \lambda) \text{ for } \lambda = \frac{\kappa^2+\xi^2}{4} \text{ and } G_\mu(x, y) = G(x, y; \mu) \text{ for } \mu = \frac{\kappa^2+\eta^2}{4}.$$

Then,

$$G_\lambda = (\mathbf{L}+\lambda)^{-1} = \sum_{s=0}^{\infty}\frac{1}{l_s+\lambda}\mathbf{E}_s, \quad G_\mu = (\mathbf{L}+\mu)^{-1} = \sum_{t=0}^{\infty}\frac{1}{l_t+\mu}\mathbf{E}_t.$$

$\mathbf{E}_s$ and $\mathbf{E}_t$ are commutative $\mathbf{E}_s\mathbf{E}_t = \mathbf{E}_t\mathbf{E}_s = \delta_{st}\mathbf{E}_s^{\,2}$, each one is idempotent $\mathbf{E}_s^{\,2} = \mathbf{E}_s$ and furthermore

$$\frac{1}{(l_s+\lambda)(l_s+\mu)} = \frac{1}{\lambda-\mu}\Big(\frac{1}{l_s+\mu} - \frac{1}{l_s+\lambda}\Big).$$

We have $G_\lambda G_\mu = G_\mu G_\lambda$ and

$$G_\lambda G_\mu = \Big(\sum_{s=0}^{\infty}\frac{\mathbf{E}_s}{l_s+\lambda}\Big)\Big(\sum_{t=0}^{\infty}\frac{\mathbf{E}_t}{l_t+\mu}\Big) = \frac{1}{\lambda-\mu}\sum_{s=0}^{\infty}\Big(\frac{\mathbf{E}_s}{l_s+\mu} - \frac{\mathbf{E}_s}{l_s+\lambda}\Big) = \frac{1}{\lambda-\mu}(G_\mu - G_\lambda).$$

q.e.d.

A second expression of $\mathcal{H}(\xi, z)$ Since $\kappa > 0$ by the assumption (1), we have an integer h greater than κ and start again from the equality

$$A^{-\kappa} = \frac{(h-1)!}{\Gamma(h-\kappa)\Gamma(\kappa)}\int_0^{+\infty}\frac{w^{h-\kappa-1}}{(A+w)^h}dw \quad (A>0) \tag{27}$$

which is true because

$$\frac{(h-\kappa)w^{h-\kappa-1}}{(A+w)^h} - \frac{hw^{h-\kappa}}{(A+w)^{h+1}} = \frac{\partial}{\partial w}\frac{w^{h-\kappa}}{(A+w)^h}$$

if $h > \kappa$. By applying (27) to $A = \cosh\theta - z$, we have

$$\mathcal{H}(\xi, z) = \frac{1}{\Gamma(h-\kappa)\Gamma(\kappa)}\int_0^{+\infty} w^{h-\kappa}dw\frac{\partial^h}{\partial z^h}\int_{-\infty}^{+\infty}\frac{e^{i\xi\theta}d\theta}{\cosh\theta - z + w}. \tag{28}$$

We make use of a formula for $0 \le b < \pi$, $\xi \in \mathbf{R}$:

$$\int_{-\infty}^{+\infty}\frac{e^{i\xi\theta}d\theta}{\cosh\theta + \cos b} = \frac{2\pi}{\sin b}\frac{\sinh[b\xi]}{\sinh[\pi\xi]}. \tag{29}$$

Proof of (29). Set $v = e^\theta$. We have then

$$\int_{-\infty}^{+\infty}\frac{e^{i\xi\theta}d\theta}{\cosh\theta + \cos b} = \int_0^{+\infty}\frac{2v^{i\xi}dv}{v^2 + 2v\cos b + 1} = \int_0^{+\infty}\frac{2v^{i\xi}dv}{(1+e^{ib}v)(1+e^{-ib}v)}$$

if b is real and independent of θ. Decompose the last fraction into two:

$$\frac{2}{(1+e^{ib}v)(1+e^{-ib}v)} = \frac{1}{i\sin b}\Big(\frac{e^{ib}}{1+e^{ib}v} - \frac{e^{-ib}}{1+e^{-ib}v}\Big).$$

Multiply $v^{i\xi}$ to both sides to have

$$\int_{-\infty}^{+\infty}\frac{e^{i\xi\theta}d\theta}{\cosh\theta+\cos b}=\frac{e^{b\xi}I_{+}-e^{-b\xi}I_{-}}{i\sin b}$$

with

$$I_{+}=\int_{0}^{+\infty}\frac{e^{ib}v^{i\xi}dv}{1+e^{ib}v}\quad\text{and}\quad I_{-}=\int_{0}^{+\infty}\frac{e^{-ib}v^{i\xi}dv}{1+e^{-ib}v}.$$

I_{+} and I_{-} come in fact from one and the same improper integral :

$$I_{+}=I_{-}=\int_{0}^{+\infty}\frac{v^{i\xi}}{1+v}dv=\Gamma(1+i\xi)\Gamma(-i\xi)=\frac{i\pi}{\sinh[\pi\xi]}.\qquad \textbf{q.e.d.}$$

Set $t=\tan(v/2)$. Then, we have

$$e^{iv/2}=\frac{1+it}{\sqrt{1+t^{2}}},\qquad dv=\frac{2dt}{t+t^{3}}.$$

By differentiation of both sides of (29) $h-1$ times with respect to z,

$$(h\text{–}1)!\int_{-\infty}^{+\infty}\frac{e^{i\xi\theta}d\theta}{(\cosh\theta-z+w)^{h}}=\frac{\partial^{h-1}}{\partial z^{h-1}}\frac{2\pi}{\sin b}\frac{\sinh[b\xi]}{\sinh[\pi\xi]},\qquad b=\arccos(w\text{–}z).\quad(30)$$

If $0<\kappa<h$, we have

$$\mathcal{H}(\xi,z)=\frac{2\pi}{\Gamma(h-\kappa)\Gamma(\kappa)}\int_{0}^{+\infty}w^{h-\kappa-1}\frac{\partial^{h-1}}{\partial z^{h-1}}\frac{1}{\sin b}\frac{\sinh[b\xi]}{\sinh[\pi\xi]}dw.\qquad(31)$$

Set $b=\arccos(w-z)$ and $z=\sum_{p=0}^{n}\sqrt{x_p y_p}\cos\phi_p$. We have finally

$$G\Big(x,y;\frac{\kappa^{2}+\xi^{2}}{4}\Big)=\frac{2^{2-\kappa}\pi}{\Gamma(\kappa)\Gamma(h-\kappa)}\int_{0}^{+\infty}t^{h-\kappa-1}dt$$

$$\times\int_{0}^{\pi}\!\!\cdots\!\int_{0}^{\pi}\frac{\partial^{h-1}}{\partial z^{h-1}}\frac{\sinh[b\xi]}{\sin b\sinh[\pi\xi]}\Big|_{z=\sum_{p=0}^{n}\sqrt{x_p y_p}\cos\phi_p}dM(\phi).\qquad(32)$$

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