On the Embedded Eigenvalue of Relativistic Schrdinger Operator *

Itaru Sasaki (Shinshu University)

J. von-Neumann and E. Wigner showed that Schrödinger operator

$$H_{\rm NR} = H_{\rm NR}(V) := -\frac{1}{2}\Delta + V(x), \qquad x \in \mathbb{R}^3 \tag{1}$$

may have a positive eigenvalue even if $V(x)$ is a smooth decaying potential[2]. Actually, they constructed the following potential and eigenfunction

$$V_{\rm NW}(x) := -32\frac{\sin|x|\left(g(|x|)^3\cos|x| - 3g(|x|)^2\sin^3|x| + g(|x|)\cos|x| + \sin^3|x|\right)}{(1+g(|x|)^2)^2}, \tag{2}$$

$$u_{\rm NW}(x) := \frac{\sin|x|}{|x|(1+g(|x|)^2)}. \tag{3}$$

where $g(|x|) = 2|x| - \sin 2|x|$. Then $H_{\rm NR}(V_{\rm NW})u_{\rm NW} = u_{\rm NW}$ holds, and $1 \in \sigma_{\rm p}(H_{\rm NR})$. Since, $V_{\rm NW}$ is a decaying potential, the essential spectrum of $H_{\rm NR}$ is $[0,\infty)$. Hence, eigenvalue one is embedded in the essential spectrum.

In this talk, we generalize the Neumann and Wigner's result to the relativistic Schrödinger operator

$$H = (-\Delta + m^2)^{1/2} - m + V, \tag{4}$$

acting on $L^2(\mathbb{R})$, where $m \geq 0$ is a mass of the particle. The operator $(-\Delta + m^2)^{1/2}$ is defined through a functional calculus. As in the non-relativistic case, if V is decaying, then essential spectrum of H is $[0,\infty)$.

We define

$$h(x) := \frac{1}{1+g(x)^2} \tag{5}$$

$$f(x) := \left(\sqrt{(-i\tfrac{d}{dx}+1)^2 + m^2} + \sqrt{(-i\tfrac{d}{dx}-1)^2 + m^2}\right)h(x). \tag{6}$$

and set

$$u(x) := f(x)\sin x \tag{7}$$

$$V(x) := \lambda - \frac{1}{u(x)}\left(\sqrt{-\tfrac{d^2}{dx^2} + m^2} - m\right)u(x), \tag{8}$$

where $\lambda := \sqrt{1+m^2} - m > 0$. The following holds.

Theorem 1. *Let H be a relativistic Schrödinger operator with V defined by (8). If $m \geq 146$,*

* Joint work with Jozsef Lörinzi (Loughborough University)

then V is smooth potential with the property that $V(x) = O(1/|x|)$ as $|x| \to \infty$. Moreover $u \in D(H)$, and λ and u satisfy the eigenvalue equation

$$Hu = \lambda u, \qquad u \in D(H). \tag{9}$$

The above result can be extended to the three dimensional case:

Theorem 2. *Assume $m \geq 146$. Let $W(x) = V(|x|)$, $x \in \mathbb{R}^3$ and define*

$$\widetilde{H} = \sqrt{-\Delta + m^2} - m + W(x) \tag{10}$$

which acts on $L^2(\mathbb{R}^3)$. Then

$$v(x) = \frac{u(|x|)}{\sqrt{4\pi}|x|}$$

belongs to $D(\tilde{H})$, and the eigenvalue equation $\tilde{H}v = \lambda v$ holds.

Remark By restoring the speed of light c as a parameter in the operator, the eigenfunction and potential constructed above converge to the expressions obtained by von Neumann and Wigner as $c \to \infty$.

The above theorem doesn't cover the massless case $m = 0$. Instead of the construction of the positive eigenvalue, we can discuss the existence of the zero energy eigenvalue.

Theorem 3. *Let us define the potential $\widetilde{V}_\nu$ and function $v_\nu(x)$.*

$$\widetilde{V}_\nu(x) := \begin{cases} -\frac{2(1-2\nu)\Gamma\left(\nu-\frac{1}{2}\right)}{(1-\nu)\sqrt{\pi}\Gamma(\nu-1)}(1+x^2)^\nu \,{}_2F_1\left(2, \frac{1}{2}+\nu; \frac{3}{2}; -x^2\right), & \text{if } \nu \neq 1 \\ -\frac{2}{1+x^2}, & \text{if } \nu = 1, \end{cases}$$

$$v_\nu(x) := \frac{x}{(1+x^2)^\nu}.$$

Then the eigenvalue equation $\sqrt{-\frac{d^2}{dx^2}}v_\nu + \widetilde{V}_\nu v_\nu = 0$ holds in the distributional sense. Moreover $\widetilde{V}_\nu$ have the following asymptotic behaviour.

$$\widetilde{V}_\nu(x) = \begin{cases} O(1/|x|), & \text{if } \frac{1}{2} < \nu < \frac{3}{2},\ \nu \neq 1 \\ O(1/|x|^2), & \text{if } \nu = 1 \\ O(\log|x|/|x|), & \text{if } \nu = \frac{3}{2} \\ O(1/|x|^{4-2\nu}), & \text{if } \frac{3}{2} < \nu < 2. \end{cases} \tag{11}$$

Remark Since $v_\nu(x) = O(1/|x|^{2\nu-1})$, $v_\nu \in L^2(\mathbb{R})$ iff $\nu > \frac{3}{4}$. Thus, the massless relativistic Schrödinger operator with $\widetilde{V}_\nu$ have a zero energy eigenvalue if $\nu > \frac{3}{4}$, and have a zero energy resonance if $\frac{1}{2} < \nu \leq \frac{3}{4}$.

References

[1] J. Lőrinczi, I. Sasaki: Embedded Eigenvalues and Neumann-Wigner Potentials for Relativistic Schrödinger Operators, preprint, 2016

[2] J. von Neumann, E. Wigner: Über merkwürdige diskrete Eigenwerte, *Z. Physik* **30**, 465-467, 1929