Abstract

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“Width of semiclassical resonances above an energy level crossing”

Let P be a 2×2 matrix-valued Schrödinger operator,

$$P = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix},$$

where $h>0$ is a small parameter and

$$P_j = h^2D_x^2 + V_j(x), \qquad D_x = \frac{1}{i}\frac{d}{dx},$$
$$W = W(x, hD_x) = r_0(x) + ir_1(x)hD_x.$$

We consider a one-dimensional model where the potentials $V_1(x)$ and $V_2(x)$ cross, and study the asymptotic distribution as $h\to 0$ of resonances near a fixed energy $E_0\in\mathbb{R}$ above the crossing level.

We impose the following assumptions on the potentials $V_1(x)$ and $V_2(x)$:

(A1) $V_1(x)$, $V_2(x)$ are real-valued on $\mathbb{R}$ and analytic in

$$\mathcal{S} := \{x\in\mathbb{C}\,;\,|\mathrm{Im}\,x| < \delta_0\langle \mathrm{Re}\,x\rangle\}.$$

(A2) $V_1(x)$, $V_2(x)$ admit limits $V_{1,\pm}$, $V_{2,\pm}$ as $\mathrm{Re}\,x\to\pm\infty$ in $\mathcal{S}$, and

$$V_{2,+} < 0 < E_0 < \min(V_{1,\pm}, V_{2,-}).$$

(A3) There exist three numbers $a<b<0<c$ such that

$$\begin{array}{lll} V_1 > E_0 \text{ on } (-\infty,a)\cup(c,+\infty), & & V_1 < E_0 \text{ on } (a,c), \\ V_2 > E_0 \text{ on } (-\infty,b), & & V_2 < E_0 \text{ on } (b,+\infty), \\ V_1'(a) < 0, & V_1'(c) > 0, & V_2'(b) < 0. \end{array}$$

(A4) Energy level crossing occurs at one point $x=0$ and

$$V_1(0) = V_2(0) = 0, \quad V_1'(0) > 0, \quad V_2'(0) < 0.$$

(A5) r_0, r_1 are bounded analytic functions on $\mathcal{S}$, are real on $\mathbb{R}$ and satisfy the ellipticity condition for W at the crossing points $(0,\pm\sqrt{E_0})$, i.e.

$$(r_0(0), r_1(0)) \neq (0,0).$$

Under these assumptions, P is a self-adjoint operator in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, and $\sigma(P)$ consists of essential spectrum near E_0. If $W \equiv 0$ (which is not the case in this talk), there would be embedded eigenvalues near E_0 associated with P_1. Under the ellipticity condition **(A5)** on W, we expect to find resonances instead near the above embedded eigenvalues. We denote the set of such resonances by $\mathrm{Res}(P)$.

For $E \in \mathbb{C}$ close enough to E_0, we define the action:

$$\mathcal{A}(E) := \int_{a(E)}^{c(E)} \sqrt{E - V_1(t)}\, dt,$$

where $a(E)$ (resp. $c(E)$) is the unique root of $V_1(x) = E$ close to a (resp. close to c). We also fix $\delta_0 > 0$ sufficiently small and $C_0 > 0$ arbitrarily large. Our aim is to investigate the resonances of P lying in the set $\mathcal{D}_h(\delta_0, C_0)$ given by

$$\mathcal{D}_h(\delta_0, C_0) := [E_0 - \delta_0, E_0 + \delta_0] - i[0, C_0 h].$$

For $h > 0$ and $k \in \mathbb{Z}$ such that $(k + \frac{1}{2})\pi h$ belongs to $\mathcal{A}([E_0 - 2\delta_0, E_0 + 2\delta_0])$, we set

$$e_k(h) := \mathcal{A}^{-1}\left((k + \frac{1}{2})\pi h\right).$$

Theorem *Under Assumptions* **(A1)**-**(A5)**, *for h small enough, one has*

$$\mathrm{Res}(P) \cap \mathcal{D}_h(\delta_0, C_0) = \{E_k(h); k \in \mathbb{Z}\} \cap \mathcal{D}_h(\delta_0, C_0),$$

where the $E_k(h)$'s are complex numbers that satisfy

$$\text{(i)} \qquad \mathrm{Re}\, E_k(h) = e_k(h) + \mathcal{O}(h^2),$$

$$\text{(ii)} \qquad \mathrm{Im}\, E_k(h) = -C(e_k(h))h^2 + \mathcal{O}(h^{7/3}),$$

uniformly as $h \to 0$. Here

$$C(E) = \frac{\pi}{\gamma \mathcal{A}'(E)} \left| r_0(0) E^{-\frac{1}{4}} \sin\left(\frac{\mathcal{B}(E)}{h} + \frac{\pi}{4}\right) + r_1(0) E^{\frac{1}{4}} \cos\left(\frac{\mathcal{B}(E)}{h} + \frac{\pi}{4}\right)\right|^2$$

with $\gamma := V_1'(0) - V_2'(0)$ and

$$\mathcal{B}(E) := \int_{b(E)}^{0} \sqrt{E - V_2(x)}dx + \int_0^{c(E)} \sqrt{E - V_1(x)}dx,$$

where $b(E)$ is the unique root of $V_2(x) = E$ close to b.

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