Nonlinear coherent states for Schrödinger equation

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Abstract: Our aim is to describe the results of a recent work in collaboration with Rémi Carles. It will be the occasion to review classical results about Schrödinger equations and coherent states. Our final goal is to describe the solutions of a nonlinear Schrödinger equation in a semi-classical regime ($\varepsilon \ll 1$):

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\,\varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon,$$

where $\psi^\varepsilon : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$. We are interested in initial data which are semi-classical wave packets (coherent states) of the form,

$$\psi^\varepsilon(0,x) = \frac{1}{\varepsilon^{d/4}} a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) e^{i\xi_0\cdot(x-x_0)/\varepsilon},$$

where $a \in \mathcal{S}(\mathbb{R}^d)$ for instance. The potential V is supposed to be smooth, real-valued, and subquadratic:

$$V \in C^\infty(\mathbb{R}^d;\mathbb{R}) \quad ; \quad \partial_x^\alpha V \in L^\infty, \quad \forall|\alpha| \geqslant 2.$$

The parameters λ and σ will be fixed in an appropriate way. In the linear case ($\lambda = 0$), it is well-known that the solution ψ^ε can be approximated by a function $\psi^\varepsilon_{app,l}$ of the same shape than $\psi^\varepsilon(0,\cdot)$ with time-dependent core $x(t)$, profile $a(t,\cdot)$ and direction of oscillations $\xi(t)$. Then, it is pertinent to ask what remains true in a nonlinear regime: $\lambda \neq 0$. We will prove that such a description can be made for special values of α and σ, and we will build an approximate solution $\psi^\varepsilon_{app,nl}$. Before proving the convergence of $\psi^\varepsilon_{app,nl}$ to ψ^ε in an appropriate functional space, we will recall well-known results about Strichartz inequalities. Finally, we will discuss the validity of our asymptotics for large times (Ehrenfest times).