Semi-classical analysis in complex geometry: Bergman kernel asymptotics for lower energy forms

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In Complex Geometry, it is crucial to be able to construct many holomorphic sections.

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$\Rightarrow$ If $R^L > 0$ then $\dim H^0(M, L^k) \approx k^n$.

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Combining this with Riemann-Roch-Hirzebruch Theorem, we can also solve GR conjecture.

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Main results, Hsiao-Marinescu, Commun. Anal. Geom. **22** (2014), No. 1, 1–108.

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Lemma (easy)

$\forall q = 0, 1, 2, \ldots, n$, *and* $\forall \lambda > 0$,

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Proof.

For every $\mu > 0$, the sequence

$$H^0_\mu(M, L^k) \to H^1_\mu(M, L^k) \to \cdots \to H^n_\mu(M, L^k) \to 0$$

is exact. □

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$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M,L^k)$$

$$=\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{\leq k^{-N_0}}(M,L^k)-\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{0<\mu\leq k^{-N_0}}(M,L^k)$$

$$\leq\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{\leq k^{-N_0}}(M,L^k)$$

$$=\sum_{j=0}^{q}(-1)^{q-j}(2\pi)^{-n}k^n\int_{M(j)}\left|\det R^L(x)\right|dv(x)+o(k^n).$$

$$\begin{aligned}
&\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M,L^k)\\
&=\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{\leq k^{-N_0}}(M,L^k)-\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{0<\mu\leq k^{-N_0}}(M,L^k)\\
&\leq\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{\leq k^{-N_0}}(M,L^k)\\
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We get strong Morse inequalities. When $q = n$, we get asymptotic Riemann-Roch-Hirzebruch Theorem.

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We obtain Bergman kernel asymptotic expansions.

Remark

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The proof of the main results base on semi-classical WKB method for Kodaira Laplacian $\Box_k^{(q)}$. We refer the readers to the paper by Hsiao-Marinescu: *Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles*, Commun. Anal. Geom. **22** (2014), No. 1, 1–108.