Low energy approximations of the Feynman path integral for Schrödinger evolution operators

Yoshihisa Miyanishi

The graduate school of Engineering and Science
Osaka University

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§1 Introduction and Motivation

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$= -ih\{(\frac{\partial H}{\partial p})\frac{\partial}{\partial x} - (\frac{\partial H}{\partial x})\frac{\partial}{\partial p}\} - \frac{1}{2}(x\frac{\partial H}{\partial x} + p\frac{\partial H}{\partial p}) + H$

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$\sum_\alpha \tilde{L}_\alpha^2 \pi^* Y_{l,m} = \frac{h^2}{2}(l(l+1) + \frac{1}{8})\pi^* Y_{l,m}$

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Problem 1. For $H = \frac{1}{2}|p|^2$, what is $\hat{H}$?

Problem 2. What is the main difficulty ?

Canonical quantization v.s. Feynman quantization

(Case 1) $M = \mathbf{R}^n$, $H(x,p) = \frac{1}{2}|p|^2 + V(x) \in C^\infty(T^*M)$

Classical mechanics	Canonical quantization	Feynman quantization
$V(x) = O(\lvert x\rvert^2)$+error. (Fujiwara theory)	$\hat{H} = -\frac{h^2}{2}\triangle + V(x)$	$\lim_{N\to\infty}[U(\frac{t}{N})]^N = \exp\left(\frac{-it}{h}\hat{H}\right)$
$V(x) = C\lvert x\rvert^n$ $(C > 0,\ n \geqq 4)$	$\hat{H} = -\frac{h^2}{2}\triangle + V(x)$	$\lim_{N\to\infty}[U(\frac{t}{N})]^N = \exp\left(\frac{-it}{h}\hat{H}\right)$?

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(Integral kernel) $e^{\frac{-it}{h}\hat{H}} f(x) = \int_{\mathbf{R}^n} K(t,x,y) f(y)\, dy$.

Classical mechanics	Orbits of CM	integral kernel
$V(x) = O(\|x\|^2)$+error.	time locally global diffeo on config. space	$K(t,x,y) \in C^\infty((0,t)\times \mathbf{R}^{2n})$
$V(x) = C\|x\|^n$ $(C > 0,\ n \geqq 4)$	infinite many small periodic curves	If $n = 1$, $K(t,x,y)$ is nowhere C^1

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Classical mechanics	geometric quantization	Feynman quantization
geodesic flow (various speeds)	prequantization exists real polalization fails! $\hat{H} = -\frac{h^2}{2}(\triangle - \frac{1}{8})$?	$\lim_{N\to\infty} [U(\frac{t}{N})]^N$ $=$ $\exp[\frac{-it}{h}\left(-\frac{h^2}{2}(\triangle - \frac{R}{6})\right)]$

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(Integral kernel) $e^{\frac{-it}{h}(-\frac{h^2}{2}\triangle - \beta R)}f(x) = \int_{\mathbf{S}^2} K(t,x,y)f(y)\,dy$.

Classical Mechanics	Orbit	Integral kernel
geodesic flow (various speeds)	infinite many small periodic curves	$K(t,x,y)$ is distribution.

Here $\triangle = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^j}(\sqrt{G}g^{ij}\frac{\partial}{\partial x^i})$ and $R = 2$ (scalar curvature).

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(Spectral geometry $\triangle + \beta R$)
・$\beta = 0$ (well-known)
・$\beta = 1/6$ (geometry of spectrum clustering)
・Geometry of prequntization

Summary (その 2)

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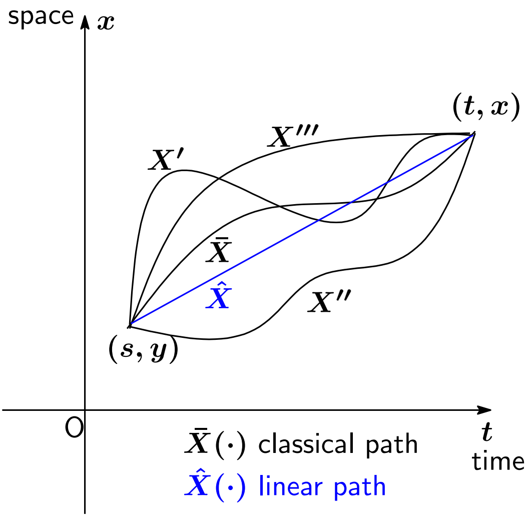
The kernel of $e^{\frac{it\triangle}{2}}$ is given by

$$K(t,x,y) = \sum_{E_j} e^{\frac{-itE_j}{2}} \overline{u_j(x)} u_j(y)$$

where $\{u_j(x)\}$ is eigenfunction expansion of $-\triangle$ and E_j are eigenvalues. The behavior of $K(t,x,y)$ is quite singular. Neverthless, when we sum a finite number of terms in E, $K_{finite}(t,x,y)$ are smooth.

§2 Feynman path integrals (heuristics)

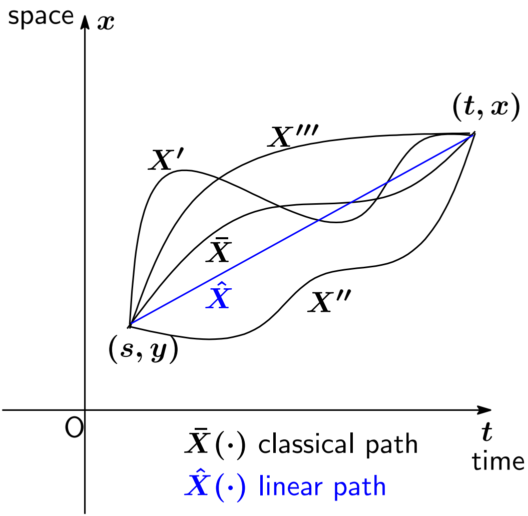


$\bar{X}(\cdot)$ classical path

$\hat{X}(\cdot)$ linear path

Action integral

$$S(s,t,x,y) = \int_s^t [\tfrac{1}{2}\dot{X}(\tau)^2 - V(X(\tau))]d\tau$$



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Ω is the path space connecting $(0, y)$ and (t, x). $\mathcal{D}[X]$ is the Lebesgue-like measure on Ω

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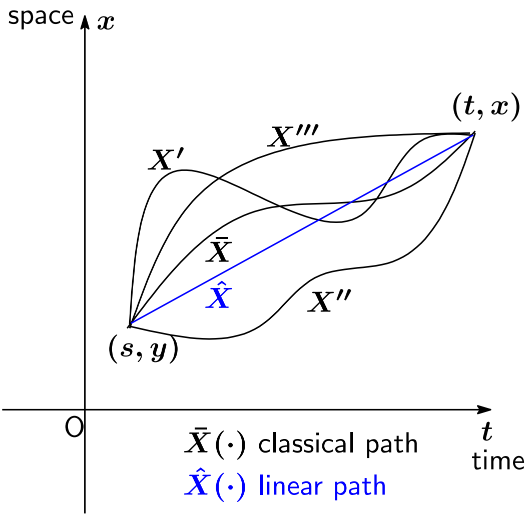

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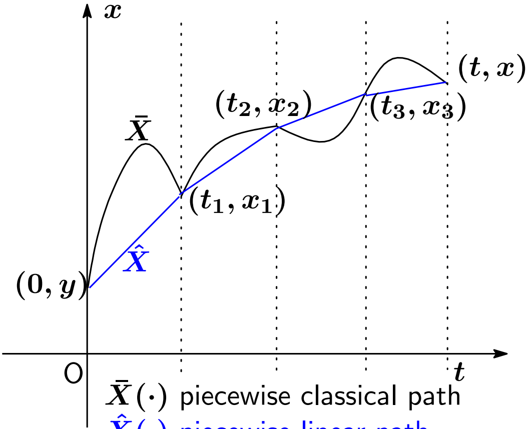
Remark. We can not construct Feynman path measure (Cameron)

Time slicing approximations (An alternative method of F.P.I.)



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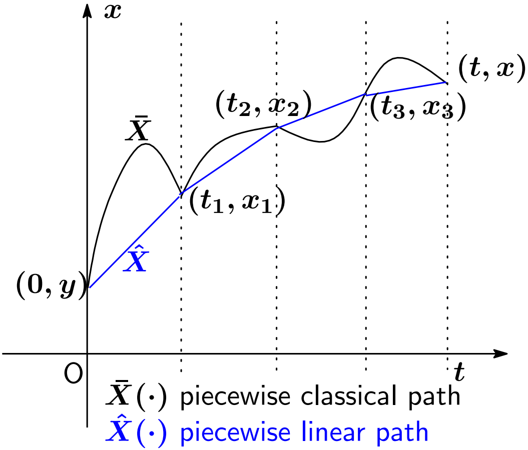

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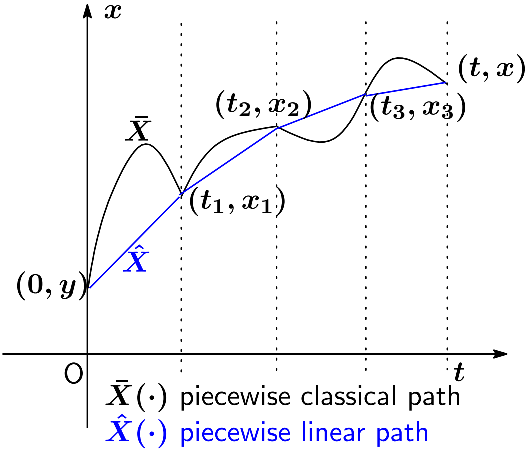

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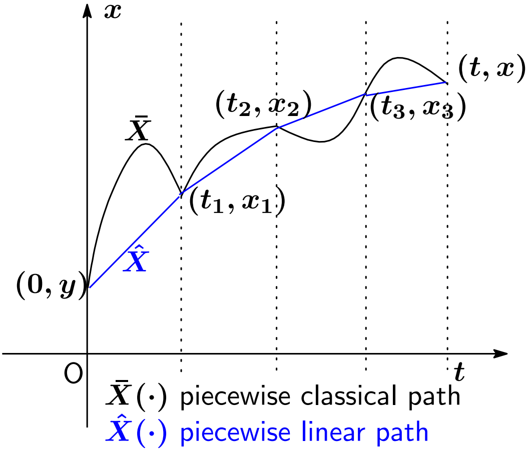

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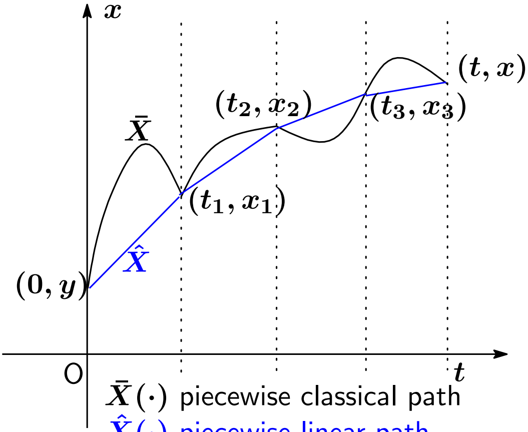

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$$\int_M a(t_j, t_{j+1}, x_j, x_{j+1}) e^{\frac{i}{\hbar} S(t_j, t_{j+1}, x_j, x_{j+1})} f(x_j) dx_j = U(t_{j+1} - t_j) f(x_{j+1})$$

(small time evolution op.)

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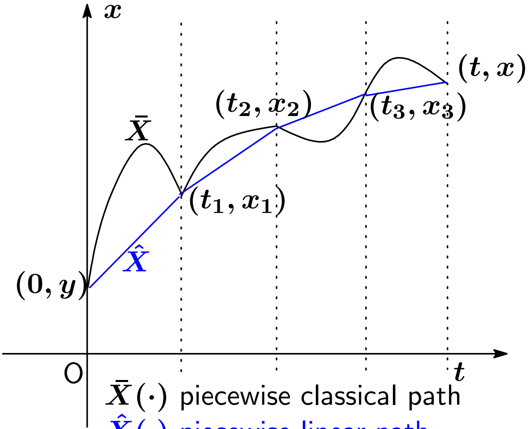

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(small time evolution op.)
Time slicing approximations are defined by

$$\int_M \cdots \int_M \prod_{j=0}^{N-1} a(t_j,t_{j+1},x_j,x_{j+1}) e^{\frac{i}{\hbar}S(t_j,t_{j+1},x_j,x_{j+1})} f(y) \prod_{j=0}^{N-1} dx_j$$
$$= [\prod_{j=0}^{N-1} U(t_{j+1}-t_j)]f(x) \to \int_\Omega e^{\frac{i}{\hbar}S(0,t,x,y)} f(y)\mathcal{D}[X] \ (N\to\infty).$$

Other different alternative definitions of Feynman path integrals

Other alternative methods for path integrals.

1. Trotter Kato forumulas.
2. Analytic continuation of Wiener measure
   by using complex Planch constant h, m or t
3. An improper integral on Hilbert spaces.
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Here, we employ the time slicing products.
to derive the curvature from action integrals.

Known results (D. Fujiwara)

Assumption $V(x) \in C^\infty(\mathbf{R}^n)$, $|\partial^\alpha V(x)| < C_\alpha$ for $|\alpha| \geqq 2$.

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For $t \neq 0$,

$$\lim_{N\to\infty} [U(\tfrac{t}{N})]^N f(x) = \exp \frac{-it}{h}[-\frac{h^2}{2}\triangle + V(x)]f(x) \quad (L^2\text{-strong})$$

Some remarks

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$\forall \varepsilon > 0$, $\exists T > 0$ s.t. $0 < t < T \Rightarrow$

$$\|U(t)f\|_{L^2} \leqq (1 + Ct)\|f\|_{L^2} \qquad \cdots (1)$$

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We consider the esitmates corresponding to (1) and (2) on the sphere.

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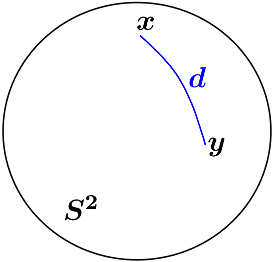

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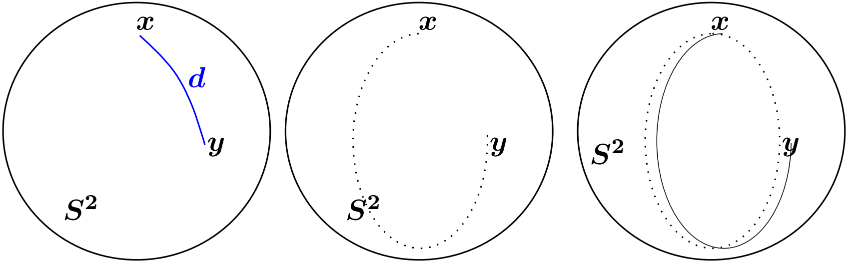

(We don't consider $d \geqq \pi$.)

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4. Van Vleck determinants on manifolds

   $$D(t,x,y) = G^{-1/2}(x) G^{-1/2}(y) \det(\partial^2 S(t,x,y)/\partial x \partial y)$$

   $\chi(d(x,y))$: cut off
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Definition (Shortest path approximations on S^2)

$$U(t)f(x) \equiv (2\pi i)^{-1} \int_{S^2} a(t,x,y) e^{iS(t,\,x,\,y)} f(y)\,dy$$

(Remark. For the simplicity, let $h = 1$.)

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$\rho(E) : L^2(S^2) \to \mathrm{L.h.}\{u_j \mid E_j \leqq E\}$: spectral projector

(Spectral projectors)

§3 Path integrals on the sphere. (Results)

[Adv. Appl. Math. Anal. (to appear)]

Theorem (operator norm)

For $t \neq 0$ and small $\varepsilon > 0$,

$$\lim_{N\to\infty} [U(t/N)]^N \rho(N^{1/3-\varepsilon}) = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right] \text{ in } L^2$$

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Corollary

Let u_j *be an eigenfunction of Laplacian. For* $t \neq 0$,

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Remark 1.

For $t \neq 0$,

$$\lim_{N\to\infty} \|[U(t/N)]^N - \exp\Big[-it\Big(-\frac{1}{2}(\triangle - \frac{R}{6})\Big)\Big]\|_{L^2} \neq 0.$$

(Time slicing products does not converge in operator norm without spectral projector,) Remark 1.

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Remark 2.

$f(x) \in G_{1/6}(S^2)$ (Gevrey class). For $t \neq 0$,

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(The convergence for low energy functions)

High energy functions cannot be captured by shortest path approximations.

§3 Path integrals on the sphere (Related results)

Let $f(x) \in C^\infty(S^2)$ and $t = \frac{8\pi m}{k} \in \mathbf{Q}$ (k and m are relatively prime.)

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$$= \int_{S^2} \sum_{l=0}^{\infty} \left(\frac{2l+1}{4\pi}\right) e^{-4\pi i\{3ml(l+1)+1\}/3k} C_l^{1/2}(\cos d(x,y))\ f(y)dy$$
$$= \left\{ e^{2\pi i m/3k} \sum_{j=0}^{2k-1} \Gamma(m,k,j) \cos \frac{2\pi j}{k} A \right\} f(x)$$

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$$= \left\{ e^{2\pi i m/3k} \sum_{j=0}^{2k-1} \Gamma(m,k,j) \cos \frac{2\pi j}{k} A \right\} f(x)$$

where $\Gamma(m,k,j) = \frac{1}{2\pi} \sum_{l=0}^{2k-1} e^{\pi i(l^2 m + lj)/k}$ is a Gaussian sum,

$A = \sqrt{-\triangle + \frac{1}{4}}$,

C_l : Gegenbauer polynomials are defined by

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{l=0}^{\infty} C_l^{1/2}(x)\ t^l.$$

We find

$$D(t,x,y)=\frac{d(x,y)}{t^2\sin d(x,y)} \quad \text{for } 0\leqq d<\pi.$$

For $\chi(d)K(t,x,y)=\chi(d)D(t,x,y)^{1/2}e^{iS}$, we obtain

$$\begin{aligned}
&\left(i\frac{\partial}{\partial t}+\frac{1}{2}\triangle_x-\frac{R}{12}\right)(\chi(d)K(t,x,y))\\
&=\left[\chi\left(\frac{d^2-\sin^2 d}{8d^2\sin^2 d}+\frac{1}{8}-\frac{R}{12}\right)+\frac{1}{2}(\triangle_x\chi)\right]K(t,x,y)\\
&\quad+\frac{\partial\chi}{\partial d}\left(\frac{\sin d-d\cos d}{2d\sin d}\right)K(t,x,y)\\
&\quad+\frac{\partial\chi}{\partial d}\left(\frac{id}{t}\right)K(t,x,y).
\end{aligned}$$

§3.1 Outline of proof

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$$\|U(t)f\|_{L^2} \leqq (1 + C_1 t)\|f\|_{L^2} + C_2 t^2 \|(-\triangle_{S^2} + 1)f\|_{L^2} \quad \cdots (3)$$

$$\|U(t)f - \exp(-it\hat{H})f\|_{L^2} \leqq \frac{C_3 t^2}{2}\|(-\triangle_{S^2} + 1)^3 f\|_{L^2} \quad \cdots (4)$$

§3.1 Outline of proof

The binomical coefficients bounds $\binom{N}{k}\frac{1}{N^k} < \frac{1}{k!}$ yields the following estimates

$$
\begin{aligned}
&\|\{e^{-it\hat{H}} - U_\chi(t/N)^n\}f(x)\|_{L^2} \\
&\quad = \| \left[e^{-it\hat{H}} - \{e^{-it\hat{H}/N}(1+\tilde{E}(t/N))\}^N \right] f(x)\|_{L^2} \\
&\quad \leqq \sum_{k=1}^{N} \binom{N}{k} \|\{e^{-i(N-k)t\hat{H}/N}\tilde{E}(t/N)^k\}f(x)\|_{L^2} \\
&\quad \leqq \sum_{k=1}^{N} \binom{N}{k} \left(\frac{\tilde{C}}{2}\right)^k \left(\frac{t}{N}\right)^{2k} \|(-\triangle+1)^{3k}f(x)\|_{L^2} \\
&\quad \leqq \sum_{k=1}^{N} \frac{1}{k!}\left(\frac{\tilde{C}t^2}{2N}\right)^k \|(-\triangle+1)^{3k}f(x)\|_{L^2}.
\end{aligned}
$$

§4 Path integrals for super quadratic potentials (The shortest linear path and low energy approximations)

Setting(1-dim super quadratic potentials)

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1. $\boldsymbol{H(x,p) = \frac{1}{2}|p|^2 + c|x|^n \in C^\infty(T^*\mathbf{R})}$, $\quad \boldsymbol{(c > 0}$, $\boldsymbol{n \geqq 4)}$.

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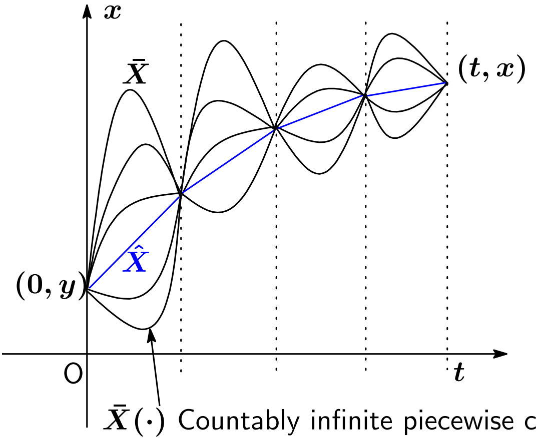

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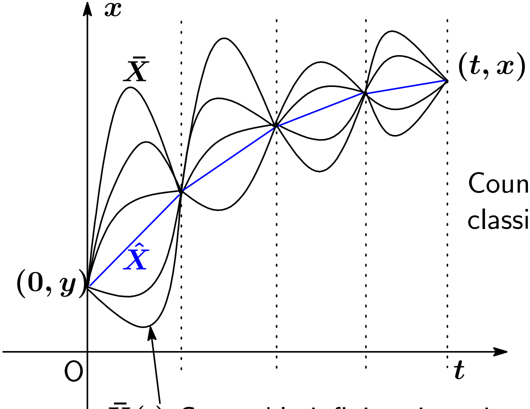

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2. $$\boldsymbol{S(t,x,y) = \int_0^t L(\hat{X}(s), \dot{\hat{X}}(s))ds = \frac{(y-x)^2}{2t} - \frac{ct}{n+1}\left(\frac{y^{n+1}-x^{n+1}}{y-x}\right)}$$
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Lemma (one path condition)

Let $C_0 < 2\sqrt{2}$ and $t > 0$. Then the classical motion satisfying $\hat{X}(s)|t|^{2/(n-2)} < C_0$ for all $0 \leqq s \leqq t$ is one at most.

3. Van Vleck determinant
$$D(t,x,y) = \partial^2 S(t,x,y)/\partial x \partial y$$
$$\chi_t(x,y) \equiv \chi(t^{2/(n-2)}x, t^{2/(n-2)}y) \text{ : cut off}$$
(bump ft. with compact support contained in $|x| < C_1, |y| < C_1$.)
$$a(t,x,y) = \chi_t(x,y) D(t,x,y)^{1/2}$$

§4 Path integrals for super quadratic potentials (The shortest linear path and low energy approximations)

Setting

Definition (Shortest path approximations)

$$U(t)f(x) \equiv (2\pi i)^{-1/2} \int_{\mathbf{R}} a(t,x,y) e^{iS(t,\, x,\, y)} f(y)\, dy$$

We consider the spectral resolutions
$-\frac{1}{2}\triangle + c|x|^n = \int_{\mathbf{R}} E d\rho(E)$: spectral resolution

Notations of Zanelli's h-small caliculus

Definition (Low energy shortest path approximations)

$$U(t,E) \equiv \rho(E)U(t)$$

(Projections onto low energy functions)

§4 Path integrals for super quadratic potentials (Results)

Theorem (Time slicing products and the strong limits)

Let $E_N = o(N^{n/2n-2})$ *and* $E_N \to \infty$ *as* $N \to \infty$. *We have*

$$s\text{-}\lim_{N\to\infty} [U(t/N, E_N)]^N f(x) = e^{-it\hat{H}} f(x) \quad \text{in } L^2(\mathrm{R})$$

Remark. If $n = 2$, $\chi_t(x, y) \equiv \chi(t^{2/(n-2)}x, t^{2/(n-2)}y)$
gives the usual oscillatory integrals.
Moreover we need not $\rho(E)$.

Remark. If we use the classical shortest paths
the improved estimates will be given.

Shell game (not rigorous!).

For small $t > 0$,

$$\int_{\Omega} e^{iS(0,t,x,x)} \mathcal{D}[X]$$

$$\sim \frac{1}{2\pi i t} \exp\left(iS(0,t,x,x) + \frac{iRt}{12}\right) \sim \sum_{\substack{l=0,1,\cdots \\ -l \le m \le l}} e^{-it\frac{l(l+1)}{2}} |Y_{l,m}(x)|^2.$$

$S(0,t,x,x) = 0$ とし, trace の実部を計算.

§7 Prospects

Shell game (not rigorous!).

For small $t > 0$,

$$\int_\Omega e^{iS(0,t,x,x)}\mathcal{D}[X]$$

$$\sim \frac{1}{2\pi it}\exp\left(iS(0,t,x,x)+\frac{iRt}{12}\right) \sim \sum_{\substack{l=0,1,\cdots\\ -l\le m\le l}} e^{-it\frac{l(l+1)}{2}}|Y_{l,m}(x)|^2.$$

$S(0,t,x,x)=0$ とし, trace の実部を計算.

$$\mathrm{Re}\left[\frac{1}{2\pi it}\int_{S^2}\exp\left(\frac{iRt}{12}\right)dx\right] \sim \mathrm{Re}\sum_{\substack{l=0,1,\cdots\\ -l\le m\le l}} e^{-it\frac{l(l+1)}{2}}$$

Shell game (not rigorous!).

For small $t > 0$,

$$\int_\Omega e^{iS(0,t,x,x)}\mathcal{D}[X]$$

$$\sim \frac{1}{2\pi it}\exp\left(iS(0,t,x,x) + \frac{iRt}{12}\right) \sim \sum_{\substack{l=0,1,\cdots \\ -l\le m\le l}} e^{-it\frac{l(l+1)}{2}}|Y_{l,m}(x)|^2.$$

$S(0,t,x,x) = 0$ とし, trace の実部を計算.

$$\mathrm{Re}\left[\frac{1}{2\pi it}\int_{S^2}\exp\left(\frac{iRt}{12}\right)dx\right] \sim \mathrm{Re}\sum_{\substack{l=0,1,\cdots \\ -l\le m\le l}} e^{-it\frac{l(l+1)}{2}}$$

$t \to 0$ とし, $\zeta(x)$ をリーマンゼータ関数とすると,

$$\begin{aligned}\int_{S^2}\left(\frac{R}{24\pi}\right)dx &= 1 + \sum_{l=1,2,\cdots}(2l+1) \\ &= 1 + 2\zeta(-1) + \zeta(0) \\ &= 1 + 2\left(\frac{-1}{12}\right) + \left(\frac{-1}{2}\right) = \frac{1}{3} \text{ (correct)}\end{aligned}$$

続き (一般の 2 次元多様体, h-small) (not rigorous!).

For small $t > 0$,

$$\int_{\Omega} e^{\frac{i}{h}S(0,t,x,x)}\mathcal{D}[X]$$

$$\sim \frac{1}{2\pi iht}\exp\left(\frac{i}{h}S(0,t,x,x) + \frac{ihRt}{12}\right) \sim \sum_{j=1,\cdots} e^{\frac{-itE_j}{h}}|u_j(x)|^2.$$

$S(0,t,x,x) = 0$ とし, trace の時間積分 ($t_1 \leqq t \leqq t_2$) を計算.

続き (一般の 2 次元多様体, h-small) (not rigorous!).

For small $t > 0$,

$$\int_\Omega e^{\frac{i}{h}S(0,t,x,x)}\mathcal{D}[X]$$

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$S(0,t,x,x)=0$ とし, trace の時間積分 ($t_1 \leqq t \leqq t_2$) を計算.

$$\int_{t_1}^{t_2}\int_M \frac{1}{2\pi iht}\exp\left(\frac{ihRt}{12}\right)dx\ dt \sim \sum_{j=1,\cdots} \frac{ih}{E_j}e^{\frac{-i(t_2-t_1)E_j}{h}}$$

§7 Some prospects

続き (一般の 2 次元多様体, h-small) (not rigorous!).

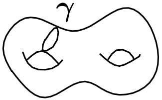
For small $t > 0$,

$$\int_\Omega e^{\frac{i}{h}S(0,t,x,x)}\mathcal{D}[X]$$

$$\sim \frac{1}{2\pi iht}\exp\left(\frac{i}{h}S(0,t,x,x)+\frac{ihRt}{12}\right) \sim \sum_{j=1,\cdots} e^{\frac{-itE_j}{h}}|u_j(x)|^2.$$

$S(0,t,x,x)=0$ とし, trace の時間積分 ($t_1 \leqq t \leqq t_2$) を計算.

$$\int_{t_1}^{t_2}\int_M \frac{1}{2\pi iht}\exp\left(\frac{ihRt}{12}\right)dx\ dt \sim \sum_{j=1,\cdots}\frac{ih}{E_j}e^{\frac{-i(t_2-t_1)E_j}{h}}$$



$$\sim \sum_{\gamma:\text{closed geodesic}} he^{iS_\gamma/h}\frac{c_\gamma e^{\frac{\pi i m_\gamma}{4}}}{\sqrt{\det(I-P_\gamma)}}\ ?$$

続き (一般の 2 次元多様体, h-small) (not rigorous!).

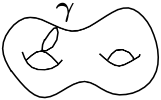
For small $t > 0$,

$$\int_\Omega e^{\frac{i}{h}S(0,t,x,x)}\mathcal{D}[X]$$

$$\sim \frac{1}{2\pi iht}\exp\left(\frac{i}{h}S(0,t,x,x) + \frac{ihRt}{12}\right) \sim \sum_{j=1,\cdots} e^{\frac{-itE_j}{h}}|u_j(x)|^2.$$

$S(0,t,x,x) = 0$ とし, trace の時間積分 ($t_1 \leqq t \leqq t_2$) を計算.

$$\int_{t_1}^{t_2}\int_M \frac{1}{2\pi iht}\exp\left(\frac{ihRt}{12}\right)dx\ dt \sim \sum_{j=1,\cdots} \frac{ih}{E_j} e^{\frac{-i(t_2-t_1)E_j}{h}}$$



$$\sim \sum_{\gamma:\text{closed geodesic}} h e^{iS_\gamma/h}\frac{c_\gamma e^{\frac{\pi i m_\gamma}{4}}}{\sqrt{\det(I-P_\gamma)}}\ ?$$

(Low energy, time local) Gutzwiller trace formula can be found in [Gu-St] (§11.5.3. p.301)

Thank you for your attention.