Semiclassical analysis in the study of hyperbolic dynamical systems

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Contents

Dynamical systems (discrete time)

Let X be a compact Riemannian manifold and $f : X \to X$ a smooth map.

- If f is hyperbolic then trajectories are complicated and unpredictable.
- Study instead the evolutions of densities under a linear operator induced by f.
- The Perron-Frobenius transfer operator M_f^* of f, or its relative the Ruelle transfer operator $M_f : \varphi \mapsto \varphi \circ f$.
- Spectral description of transfer operators can be used to obtain other specific properties of the dynamics: decay of time correlation functions, central limit theorem, mixing etc. (Ruelle, Bowen, Fried, Rugh and others).
- Spectral approach has in the last ten years been improved through the construction of functional spaces adapted to the dynamics (Blank, Gouzel, Keller, Liverani, Baladi, Tsujii and others).

Spectral gap

What one would like: spectral gap of $M_f : \mathcal{H} \to \mathcal{H}$, i.e. the moduli of the largest and second largest eigenvalues (counting multiplicities) are separated by a positive distance.

- Exponential mixing is equivalent to 1 being the only eigenvalue on the unit circle.

The spectral study of the discrete spectrum of M_f (the *Ruelle resonances*) can be treated as the quantum resonances in open quantum systems.

Symplectic properties of the dynamics on T^*X

$\to$ spectral properties of M_f

$\to$ long term behavior of the dynamics.

Semiclassical approach

For hyperbolic dynamics, the study of transfer operators is naturally understood in the semiclassical framework. Ideas appear in

- 2007 – Baladi, Tsujii.
- 2008 – Baladi, Tsujii.

Formalized in

- 2006 – Faure, Roy.
- 2008 – Faure, Roy, Sjöstrand.

Other studies have appeared afterwards (Arnoldi, Dyatlov, Faure, Sjöstrand, Tsujii, Weich, Zworski and others).

- 2011 – Faure, *Semiclassical origin of the spectral gap for transfer operators of a partially expanding map*, Nonlinearlity **24**.

Semiclassical approach

Find function space $\mathcal{H}$ such that $M_f : \mathcal{H} \to \mathcal{H}$ has a spectral gap.
Note

- $M_f : \varphi \mapsto \varphi \circ f$ is an FIO,

$$M_f\varphi(x) = \frac{1}{(2\pi)^n} \iint e^{i(f(x)-y)\xi}\varphi(y)dyd\xi.$$

  The associated symplectic map is the lift of f^{-1},

$$F : (f(x), \eta) \mapsto (x, {}^t f'(x)\eta). \tag{1}$$

- Usual Sobolev space $H^m = \mathrm{Op}\left(\langle\xi\rangle^m\right)^{-1}(L^2)$.

So: construct a symbol $a_m \in S^m$ appropriately adapted to the symplectic map (1) and set $\mathcal{H}^m = \mathrm{Op}\left(a_m\right)^{-1}(L^2)$. (Anisotropic Sobolev space.) Study $M_f : \mathcal{H}^m \to \mathcal{H}^m$.

Proving the existence of a spectral gap I

Set $A_m = \mathrm{Op}\,(a_m)$.

- The commutative diagram

$$\begin{array}{ccc} L^2 & \overset{Q_m}{\to} & L^2 \\ \downarrow A_m^{-1} & \circlearrowleft & \downarrow A_m^{-1} \\ \mathcal{H}^m & \overset{M_f}{\to} & \mathcal{H}^m \end{array}$$

  shows that $M_f : \mathcal{H}^m \to \mathcal{H}^m$ is unitarily equivalent to

$$Q_m = A_m M_f A_m^{-1} : L^2 \to L^2.$$

- Use Egorov's theorem to show that $Q_m^* Q_m$ is a pseudo-differential operator and calculate the principal symbol q_m.
- Use the L^2 continuity theorem to get $Q_m^* Q_m = R_\varepsilon + K_\varepsilon$ where K_ε is smoothing and

$$\|R_\varepsilon\|_{\mathcal{L}(L^2)} \leq \limsup_{|\xi|\to\infty} \sup_x |q_m(x,\xi)| + \varepsilon, \quad \varepsilon > 0.$$

Proving the existence of a spectral gap II

Now,

$$Q_m^* Q_m = A_m^{-1} M_f^* A_m^2 M_f A_m^{-1}$$

so

$$q_m(x,\xi) = \frac{a_m^2 \circ F(x,\xi)}{a_m^2(x,\xi)}.$$

Since a_m is strictly decreasing along trajectories of F we have $q_m < 1$ as $|\xi| \to \infty$. Hence, $Q_m^* Q_m$ has essential spectral radius < 1, and peripheral spectrum described by the compact operator K_ε.

Deduce the same representation for Q_m using polar decomposition and the spectral theorem. Use the unitary equivalence of $Q_m : L^2 \to L^2$ and $M_f : \mathcal{H}^m \to \mathcal{H}^m$ to get the same for M_f.

The model

Let $\mathbb{T}^2 = S^1 \times S^1$ where $S^1 = \mathbb{R}/\mathbb{Z}$. Define a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$f : \begin{pmatrix} x \\ s \end{pmatrix} \mapsto \begin{pmatrix} 2x \mod 1 \\ s + \tau(x) \mod 1 \end{pmatrix},$$

where $E(x) := 2x \mod 1$ is *uniformly expanding*,

$$\min_x \frac{dE(x)}{dx} > 1,$$

and $\tau : S^1 \to \mathbb{R}$ is C^∞. Both $E : S^1 \to S^1$ and $f : \mathbb{T}^2 \to \mathbb{T}^2$ are $2:1$ maps.

- Assume that f is *partially captive*. (τ cannot be cohomologous to a constant.)

Semiclassical parameter introduced by Fourier decomposition

- With f as above, define the Ruelle transfer operator $M_f : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ by

$$M_f \varphi(x,s) = \varphi(f(x,s)), \quad \varphi \in L^2(\mathbb{T}^2).$$

- M_f preserves the following decomposition in Fourier modes:

$$L^2(\mathbb{T}^2) = \bigoplus_{\nu \in \mathbb{Z}} \mathcal{H}_\nu, \quad \mathcal{H}_\nu = \{(x,s) \mapsto \varphi(x) e^{2i\pi\nu s} : \varphi \in L^2(S^1)\}.$$

  Note that for $\psi(x,s) = \varphi(x) e^{2i\pi\nu s} \in \mathcal{H}_\nu$, we have that

$$M_f \psi(x,s) = \varphi(E(x)) e^{i\nu\tau(x)} e^{2i\pi\nu s}.$$

- Let $M_\nu : L^2(S^1) \to L^2(S^2)$ denote the operator M_f restricted to $\mathcal{H}_\nu$ (with identification of $\mathcal{H}_\nu$ with $L^2(S^1)$). Explicitly,

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For fixed $\nu \in \mathbb{Z}$,

$$M_\nu \varphi(x) = \varphi(E(x)) e^{i\nu\tau(x)}$$

is a Fourier integral operator associated to the canonical relation

$$F(y, \eta) = \{(x, E'(x)\eta) : y = E(x)\} \subset T^*(S^1).$$

When $\nu \in \mathbb{Z}^+$ is considered as a semiclassical parameter $h = 1/\nu$,

$$M_{\pm\nu} \varphi(x) = \varphi(E(x)) e^{\pm i\nu\tau(x)}$$

is a semiclassical Fourier integral operator associated to the canonical relation

$$F(y, \eta) = \{(x, E'(x)\eta \pm \tau_0'(x)) : y = E(x)\} \subset T^*(S^1).$$

Since $E : S^1 \to S^1$ is a $2:1$ map, there are precisely 2 distinct elements belonging to the set in the right-hand side. Hence the map $F : T^*S^1 \to T^*S^1$ is 2 valued.

- Partial captivity essentially means that almost all trajectories escape toward infinity.

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When $\nu \in \mathbb{Z}^+$ is considered as a semiclassical parameter $h = 1/\nu$,

$$M_{\pm\nu} \varphi(x) = \varphi(E(x)) e^{\pm i\tau(x)/h}$$

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Since $E : S^1 \to S^1$ is a $2:1$ map, there are precisely 2 distinct elements belonging to the set in the right-hand side. Hence the map $F : T^*S^1 \to T^*S^1$ is 2 valued.

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Discrete spectrum

Recall $\min_x E'(x) = 2$.

Theorem (Ruelle 1986, Faure 2011)

Let $m < 0$. For any $\nu \in \mathbb{Z}$, the operator

$$M_\nu : H^m(S^1) \to H^m(S^1)$$

is a bounded operator which can be written

$$M_\nu = R_\nu + K_\nu,$$

where K_ν is a compact operator, and

$$\|R_\nu\|_{H^m} \leq 2^m \to 0, \quad m \to -\infty.$$

Spectral gap in the semiclassical limit

Let $r(M_\nu)$ be the spectral radius of the operator $M_\nu : H^m(S^1) \to H^m(S^1)$.

Theorem (Tsujii 2008, Faure 2011)

Assume that f is partially captive. Then $r(M_\nu)$ does not depend on m and in the semiclassical limit $\nu \to \infty$,

$$r(M_\nu) \le 2^{-\frac{1}{2}} + o(1).$$

More precisely, for any $\rho > 2^{-1/2}$, there are positive constants m_0 and ν_0, such that for any $m \ge m_0$, $\nu \ge \nu_0$ and $n \in \mathbb{N}$ we have

$$\|M_\nu^n\|_{\mathcal{L}(H_\nu^m)} \le c\rho^n,$$

where H_ν^m is the Sobolev space $H^m(S^1)$ equipped with the norm

$$\|\varphi\|_{H_\nu^m}^2 = \sum_{\xi \in 2\pi\mathbb{Z}} |(1 + (\xi/\nu)^2)^m \hat{\varphi}(\xi)|^2.$$