Berezin-Toeplitz Operators in the Spectral and Scattering Theory of Magnetic Quantum Hamiltonians

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The two first sections are mainly based on:

J.Avron, I.Herbst, B.Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J. **45** (1978), 847-883,

H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer-Verlag, Berlin, 1987,

A.Mohamed, G.D.Raikov, *On the spectral theory of the Schrödinger operator with electromagnetic potential*, In: Adv. Part. Diff. Equat. **5**, (1994) Akademie-Verlag, 298-390,

M.Dimassi, G.D.Raikov, *Spectral asymptotics for quantum Hamiltonians in strong magnetic fields*, Cubo Matemática Educacional, **3** (2001), 317-391.

1. Magnetic Schrödinger operators

1.1. Basic definitions and properties

Let $d \geq 2$,

$$A = (A_1, \dots, A_d) \in L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d),$$

$$V_+ \in L^1_{\text{loc}}(\mathbb{R}^d), \quad V_+ \geq 0.$$

Then the quadratic form

$$h_+[u] := \int_{\mathbb{R}^d} \left(|i\nabla u + Au|^2 + V_+|u|^2 \right) dx \geq 0,$$

defined on $C_0^\infty(\mathbb{R}^d)$, is closable in $L^2(\mathbb{R}^d)$. Let $H(A, V_+)$ be the self-adjoint operator generated by the closure of h_+. The operator

$$H(A, V_+) = (-i\nabla - A)^2 + V_+$$

is the *Schrödinger operator* with magnetic potential A and electric potential V_+. Note that

$$H(0, V_+) = -\Delta + V_+.$$

If $A \in L^4_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, $\operatorname{div} A \in L^2_{\text{loc}}(\mathbb{R}^d)$, and $V_+ \in L^2_{\text{loc}}(\mathbb{R}^d)$, then $H(A, V_+)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

1.2. Diamagnetic inequality

Some notations:

Let $\mathcal{H}_j$, $j = 1, 2$, be two (separable) Hilbert spaces. Then $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ (resp., $S_\infty(\mathcal{H}_1, \mathcal{H}_2)$) is the space of bounded (resp., compact) linear operators $T : \mathcal{H}_1 \to \mathcal{H}_2$, and $S_p(\mathcal{H}_1, \mathcal{H}_2)$, $p \in [1, \infty)$, is the pth Schatten-von Neumann class of operators $T \in S_\infty(\mathcal{H}_1, \mathcal{H}_2)$ for which

$$\|T\|_p := \left(\mathrm{Tr}\,(T^*T)^{p/2}\right)^{1/p} < \infty.$$

If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, then we will write $\mathcal{L}(\mathcal{H})$ and $S_p(\mathcal{H})$, $p \in [1, \infty]$. Occasionally, we will drop $\mathcal{H}$ in these notations.

Let $T, Q \in \mathcal{L}(L^2(M, d\mu))$ where M is a space with measure μ . We write $T \overset{\leq}{\cdot} Q$ if

$$|(Tu)(x)| \leq (Q|u|)(x), \quad u \in L^2(M, d\mu),$$

for almost every $x \in M$.

Theorem 1. (Dodds-Fremlin-Pitt)
Let $T \overset{\leq}{\cdot} Q$.
(i) *If $Q \in S_\infty(L^2(M, d\mu))$, then $T \in S_\infty(L^2(M, d\mu))$.*
(ii) *If $Q \in S_{2n}(L^2(M, d\mu))$ with $n \in \mathbb{N}$, then $T \in S_{2n}(L^2(M, d\mu))$.*

The second part of the theorem is false if we replace $2n$ with $n \in \mathbb{N}$ by $p \in [1, \infty) \setminus 2\mathbb{N}$.

Theorem 2. *Let*

$$A \in L^2_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d),\ V_+ \in L^1_{\text{loc}}(\mathbb{R}^d),\ V_+ \geq 0. \tag{1}$$

Then the diamagnetic inequality

$$\exp\left(-tH(A,V_+)\right) \overset{\leq}{\cdot} \exp\left(-t(-\Delta + V_+)\right) \overset{\leq}{\cdot} \exp\left(t\Delta\right)$$

holds true for each $t \geq 0$.

Idea of the proof: If A and V are regular, and div $A = 0$, then the operator $\exp\left(-tH(A,V_+)\right)$ with $t > 0$ admits an integral kernel

$$\mathcal{K}_{A,V_+}(\mathbf{x},\mathbf{y};t) =$$

$$\int e^{-i\int_0^t A(\omega(s))\cdot d\omega(s)} e^{-\int_0^t V_+(\omega(s))ds} dE_{0,\mathbf{x};t,\mathbf{y}}(\omega(s))$$

where $E_{0,\mathbf{x};t,\mathbf{y}}(\omega(s))$ is the conditional Wiener measure on set of paths

$$\left\{\omega \in C([0,t];\mathbb{R}^d) \,|\, \omega(0) = \mathbf{x},\ \omega(t) = \mathbf{y}\right\}.$$

In particular, we have

$$|\mathcal{K}_{A,V_+}(\mathbf{x},\mathbf{y};t)| \leq \mathcal{K}_{0,V_+}(\mathbf{x},\mathbf{y};t) \leq \mathcal{K}_{0,0}(\mathbf{x},\mathbf{y};t)$$

for $t > 0$, $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}$.

Corollary 1. *Assume* (1). *Then*

$$(H(A,V_+)+E)^{-\gamma} \lneqq (-\Delta+V_++E)^{-\gamma} \lneqq$$

$$(-\Delta+E)^{-\gamma},\ E>0,\ \gamma>0.$$

Proof: If $Q=Q^*\geq 0$, and $E>0$, $\gamma>0$, then

$$(Q+E)^{-\gamma}=\frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-tQ}e^{-tE}t^{\gamma-1}\,dt.$$

Corollary 2. *Assume* (1). *Let $V_-\geq 0$ be a measurable function over $\mathbb{R}^d$. If the multiplier by V_- is Δ-bounded* (*resp., $-\Delta$-form-bounded*) *with relative bound a, then V_- is $H(A,V_+)$-bounded* (*resp., $H(A,V_+)$-form-bounded*) *with relative bound* (*resp., relative form-bound*) *at most a.*

Idea of the proof:

(i) The $H(A, V_+)$-relative bound of V_- is equal to

$$\lim_{E\to\infty} \|V_-(H(A, V_+) + E)^{-1}\|.$$

(ii) The $H(A, V_+)$-relative form-bound of V_- is equal to

$$\lim_{E\to\infty} \|V_-^{1/2}(H(A, V_+) + E)^{-1/2}\|^2.$$

Theorem 3. *Assume* (1). *Let the multiplier by the measurable function* $V_- : \mathbb{R}^d \to [0, \infty)$ *be* Δ*-bounded* (*resp.*, $-\Delta$*-form-bounded*) *with relative bound* (*resp., relative form bound*) *smaller than one. Set* $V = V_+ - V_-$. *Then the operator sum* (*resp., form sum*)

$$H(A, V) := H(A, V_+) - V_- = (-i\nabla - A)^2 + V$$

is self-adjoint in $L^2(\mathbb{R}^d)$. *Moreover we have*

$$\exp(-tH(A, V)) \overset{\leq}{\cdot} \exp(-t(-\Delta + V)), \quad t \geq 0.$$

Exercise: Prove that

$$\inf \sigma(H(A, V)) \geq \inf \sigma(-\Delta + V),$$

$$\inf \sigma_{\text{ess}}(H(A, V)) \geq \inf \sigma_{\text{ess}}(-\Delta + V),$$

where $\sigma(T)$ (resp., $\sigma_{\text{ess}}(T)$) denotes the spectrum (resp., the essential spectrum) of the operator $T = T^*$.

1.3. Magnetic field
Assume $A \in C^1(\mathbb{R}^d; \mathbb{R}^d)$. Let

$$\mathcal{A} := \sum_{j=1}^{d} A_j dx_j, \quad \mathcal{B} := d\,\mathcal{A}.$$

We have

$$\mathcal{B} = \sum_{1 \leq j < k \leq d} B_{jk}\, dx_j \wedge dx_k$$

with

$$B_{jk} := \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}, \quad j, k = 1, \ldots, d.$$

Then $B := \left\{B_{jk}\right\}_{j,k=1}^{d}$ is *the magnetic field* generated by the magnetic potential A. Set

$$\Pi_j(A) := -i\frac{\partial}{\partial x_j} - A_j, \quad j = 1, \ldots, d,$$

so that

$$H(A, 0) = \sum_{j=1}^{d} \Pi_j(A)^2.$$

We have

$$B_{jk} = -i[\Pi_j(A), \Pi_k(A)], \quad j, k = 1, \ldots, d.$$

1.4. Gauge invariance

Assume that the magnetic potentials $A^{(j)}$, $j = 1, 2$, generate the same magnetic field, i.e. $d\,(\mathcal{A}^{(1)} - \mathcal{A}^{(2)}) = 0$. Then there exists a function $\Phi : \mathbb{R}^d \to \mathbb{R}$ such that

$$A^{(1)} - A^{(2)} = \nabla\Phi$$

(since $\mathbb{R}^d$ is simply connected, all closed 1-forms are exact). Hence,

$$e^{-i\Phi}\Pi_j(A^{(1)})e^{i\Phi} = \Pi_j(A^{(2)}), \quad j = 1, \dots, d,$$

and

$$e^{-i\Phi}H(A^{(1)}, V)e^{i\Phi} = H(A^{(2)}, V),$$

i.e. the operator $H(A^{(1)}, V)$ and $H(A^{(2)}, V)$ are unitarily equivalent under the gauge transform $u \mapsto e^{i\Phi}u$.

Remarks: (i) Performing a gauge transform, we can achieve that $\operatorname{div} A = 0$.

(ii) The operators $H(A, V)$ and $H(-A, V)$ are anti-unitarily equivalent under the complex conjugation.

1.5. Constant magnetic field

Let $d \geq 2$. Assume that $B \neq 0$ is constant with respect to $x \in \mathbb{R}^d$. Then we will call

$$H_L := H(A, 0)$$

the Landau Hamiltonian. Set

$$2m := \dim \operatorname{Ran} B, \quad n := \dim \operatorname{Ker} B$$

so that $d = 2m + n$. Let $b_1 \geq \ldots b_m > 0$ be such numbers that the non-zero eigenvalues of B counted with multiplicities coincide with $\pm ib_j$, $j = 1, \ldots, m$. If $n > 0$ (resp., if $n = 0$), then in $\mathbb{R}^d$ there exist Cartesian coordinates (x, y, w) with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $w \in \mathbb{R}^n$ (resp., (x, y) with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$), such that the 2-form $\mathcal{B}$ can be written as

$$\mathcal{B} = \sum_{j=1}^{m} b_j \, dx_j \wedge dy_j.$$

Then the Landau Hamiltonian is

$$\sum_{j=1}^{m}\left(\left(-i\frac{\partial}{\partial x_j}+\frac{b_j y_j}{2}\right)^2+\left(-i\frac{\partial}{\partial y_j}-\frac{b_j x_j}{2}\right)^2\right)$$

$$-\sum_{\ell=1}^{n}\frac{\partial^2}{\partial w_\ell^2};$$

if $n = 0$, the sum with respect ℓ should be omitted.

If $n = 0$, then $\sigma(H_L)$ consists of isolated eigenvalues of infinite multiplicity, called *Landau levels*. If $n > 0$, then $\sigma(H_L) = [\Lambda_0, \infty)$ where

$$\Lambda_0 = \sum_{j=1}^{m} b_j$$

is the lowest Landau level. Moreover, $\sigma(H_L)$ is absolutely continuous (a.c.), and the higher Landau levels are embedded *spectral thresholds*. For simplicity, we will consider only the cases $d = 2$ (i.e. $m = 1$ and $n = 0$) and $d = 3$ (i.e. $m = 1$ and $n = 1$).

1.6. Spectrum of the 2D Landau Hamiltonian

Assume $d = 2$ (i.e. $m = 1$ and $n = 0$). Set $b := b_1 > 0$. Then the Landau Hamiltonian is

$$H_L = \left(-i\frac{\partial}{\partial x} + \frac{by}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{bx}{2}\right)^2 =$$

$$a^* a + b$$

where

$$a^* := -2ie^{\varphi}\frac{\partial}{\partial z}e^{-\varphi}, \quad z = x + iy,$$

is *the magnetic creation operator*,

$$a := -2ie^{-\varphi}\frac{\partial}{\partial \overline{z}}e^{\varphi}, \quad \overline{z} = x - iy,$$

is *the magnetic annihilation operator*, and

$$\varphi(x, y) := \frac{b(x^2 + y^2)}{4}.$$

We have

$$[a, a^*] = 2b.$$

Therefore,

$$\sigma(H_L) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}$$

with

$$\Lambda_q := b(2q+1), \quad q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}.$$

Moreover,

$$\text{Ker}\,(H_L - \Lambda_q) = (a^*)^q \text{Ker}\, a, \quad q \in \mathbb{Z}_+,$$

and

$$\text{Ker}\, a = \left\{ u \in L^2(\mathbb{R}^2) \mid u = g e^{-\varphi},\ \frac{\partial g}{\partial \overline{z}} = 0 \right\}$$

is the *Fock-Bargmann-Segal* space. Hence, in particular,

$$\dim \text{Ker}\,(H_L - \Lambda_q) = \infty, \quad q \in \mathbb{Z}_+.$$

Let $p_q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $q \in \mathbb{Z}_+$, be the orthogonal projection onto $\text{Ker}\,(H_L - \Lambda_q)$.

Then p_q admits an integral kernel

$$K_{q,b}(x,y;x',y') =$$

$$\frac{b}{2\pi}\mathsf{L}_q\left(\frac{b}{2}\left((x-x')^2+(y-y')^2\right)\right)\times$$

$$\exp\left\{-\frac{b}{4}[(x-x')^2+(y-y')^2+2i(xy'-yx')]\right\}$$

where $(x,y),(x',y') \in \mathbb{R}^2$, and

$$\mathsf{L}_q(t) :=$$

$$\frac{1}{q!}e^t\frac{d^q}{dt^q}\left(t^q e^{-t}\right) = \sum_{j=0}^{q}\binom{q}{j}\frac{1}{j!}(-1)^j t^j,$$

$$t \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

are the Laguerre polynomials.

In particular,

$$K_{q,b}(x,y;x,y) = \frac{b}{2\pi}, \quad (x,y) \in \mathbb{R}^2.$$

1.7. Spectrum of the 3D Landau Hamiltonian

Assume $d = 3$ (i.e. $m = 1$ and $n = 1$). Then the Landau Hamiltonian is

$$H_L = \left(-i\frac{\partial}{\partial x} + \frac{by}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{bx}{2}\right)^2 - \frac{\partial^2}{\partial w^2}.$$

Since $d = 3$, we can introduce *the magnetic-field vector* $\mathbf{B} := \operatorname{curl} A$. If $A = \frac{b}{2}(-y, x, 0)$ as above, then $\mathbf{B} = (0, 0, b)$.

That is why, if $\mathbf{x} = (x, y, w) \in \mathbb{R}^3$, we will write occasionally $\mathbf{x} = (x_\perp, x_{||})$ where $x_\perp = (x, y) \in \mathbb{R}^2$ are the variables on the plane perpendicular to $\mathbf{B}$, while $x_{||} = w \in \mathbb{R}$ is the variable along $\mathbf{B}$.

Using $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R})$, we can write

$$H_L = H_\perp \otimes I_{||} + I_\perp \otimes H_{||}$$

where $H_\perp$ is the 2D Landau Hamiltonian,

$$H_{||} := -\frac{d^2}{dx_{||}^2},$$

and $I_\perp, I_{||}$ are the respective identities.

Since $\sigma(H_{||}) = [0, \infty)$, and $\sigma(H_{||})$ is a.c., we have

$$\sigma(H_L) = \bigcup_{q=0}^{\infty} [\Lambda_q, \infty) = [\Lambda_0, \infty) = [b, \infty),$$

$\sigma(H_L)$ is a.c., and Λ_q, $q \geq 1$ are embedded spectral thresholds.

2. Pauli operators

2.1. Basic definitions

Let $d = 2, 3$. Assume

$$A \in C^1(\mathbb{R}^d; \mathbb{R}^d),\ B_{jk} \in L^\infty(\mathbb{R}^d),\ j, k = 1, \dots, d,$$

$$V \in L^\infty(\mathbb{R}^d; \mathbb{R}).$$

Introduce the Pauli matrices

$$\widehat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \widehat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \widehat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$\widehat{\sigma}_j = \widehat{\sigma}_j^*, \quad \widehat{\sigma}_j\widehat{\sigma}_k + \widehat{\sigma}_k\widehat{\sigma}_j = 2\delta_{jk}, \quad j, k = 1, 2, 3.$$

Let I_2 be the identity in $\mathbb{C}^2$, and

$$M(A,V) := \left(\sum_{j=1}^{d} \widehat{\sigma}_j \Pi_j(A) \right)^2 + V I_2$$

be the Pauli operator, self-adjoint in $L^2(\mathbb{R}^d;\mathbb{C}^2)$ and essentially self-adjoint on $C_0^\infty(\mathbb{R}^d;\mathbb{C}^2)$.

The operator $M(A,V)$ is the quantum Hamiltonian of a non-relativistic quantum particle of spin $\frac{1}{2}$, subject to an electromagnetic potential (A,V).

Similarly to H, the Pauli operator M is gauge invariant. In contrast to H, there is no diamagnetic inequality for M.

More information on the Pauli operator in arbitrary dimension could be found, e.g. in I.Shigekawa, *Spectral properties of a Schrödinger operators with magnetic fields for a spin $\frac{1}{2}$ particle*, J.Funct.Anal. **101** (1991), 255-285.

2.2. 2D Pauli operator

Let $d = 2$. Set $b := B_{12}$. Then

$$M(A, 0) = H(A, 0)I_2 - b\hat{\sigma}_3.$$

Let φ be a solution of the Poisson equation

$$\Delta\varphi = b.$$

Then, up to a gauge transform, we have

$$A = \left(-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x}\right),$$

and

$$M(A, 0) = \begin{pmatrix} m^- & 0 \\ 0 & m^+ \end{pmatrix} := \begin{pmatrix} a^*a & 0 \\ 0 & aa^* \end{pmatrix},$$

where, as earlier,

$$a := -2ie^{-\varphi}\frac{\partial}{\partial\overline{z}}\, e^{\varphi},\ a^* := -2ie^{\varphi}\frac{\partial}{\partial z}\, e^{-\varphi}.$$

Therefore,

$$\text{Ker } M(A,0) =$$

$$\left\{(u_1,u_2)\in L^2(\mathbb{R}^d,\mathbb{C}^2)\,|\,u_1\in \text{Ker } a,\; u_2\in \text{Ker } a^*\right\},$$

and

$$\text{Ker } a = \left\{u\in L^2(\mathbb{R}^2)|u = ge^{-\varphi},\; \frac{\partial g}{\partial \overline{z}} = 0\right\},$$

$$\text{Ker } a^* = \left\{u\in L^2(\mathbb{R}^2)|u = he^{\varphi},\; \frac{\partial h}{\partial z} = 0\right\}.$$

Note that we have again

$$[a,a^*] = 2b,$$

but b is no longer assumed to be constant.

Exercise: Calculate $\dim \operatorname{Ker} M(A,0)$ if:

(i) $b \in C_0^\infty(\mathbb{R}^2)$;

(ii) b is periodic, i.e.

$$b(\mathbf{x}) = \sum_{k \in \mathbb{Z}^2} b_k e^{ik\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^2,$$

with, say, $\{b_k\}_{k \in \mathbb{Z}^2} \in \ell^1(\mathbb{Z}^2)$.

2.3. 3D magnetic Pauli operator

Let $d = 3$. Then we have

$$M(A,0) = H(A,0)I_2 - \sum_{j=1}^{3} B_j \hat{\sigma}_j.$$

Let us consider *magnetic fields of constant direction* $\mathbf{B} = (0,0,b)$. Due to the closedness of $\mathcal{B}$, we have

$$b = b(x_\perp), \quad x_\perp = (x,y) \in \mathbb{R}^2.$$

As above, let φ be a solution of

$$\Delta\varphi(x_\perp) = b(x_\perp).$$

Then we can choose

$$A = \left(-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x}, 0\right).$$

Hence,

$$M(A,0) = \begin{pmatrix} M^- & 0 \\ 0 & M^+ \end{pmatrix}$$

where

$$M^\pm = (-i\nabla - A)^2 \pm b = M_\perp^\pm \otimes I_{\shortparallel} + I_\perp \otimes M_{\shortparallel},$$

$M_\perp^\pm := m^\pm$, and $M_{\shortparallel} = H_{\shortparallel} = -\frac{d^2}{dx_{\shortparallel}^2}$.

3. Berezin-Toeplitz operators

3.1. Motivation: eigenvalue asymptotics for the perturbed 2D Landau Hamiltonian

Let $d = 2$, $b = \text{const.} > 0$,

$$V \in L^\infty(\mathbb{R}^2; \mathbb{R}), \qquad \lim_{|\mathbf{x}| \to \infty} V(\mathbf{x}) = 0.$$

Then

$$|V|^{1/2} H_L^{-1/2} \in S_\infty(L^2(\mathbb{R}^2)),$$

and

$$\sigma_{\text{ess}}(H_L + V) = \sigma_{\text{ess}}(H_L) = \sigma(H_L) = \bigcup_{j=0}^{\infty} \left\{\Lambda_j\right\}.$$

However, generically, the Landau levels are accumulation points of $\sigma_{\text{disc}}(H_L + V)$.

Assume $V \geq 0$. Then the discrete eigenvalues of $H_L + V$ (resp., of $H_L - V$) may accumulate to the Landau levels only from above (resp., only from below).

Notation: If $F_j(V;\lambda)$, $j = 1, 2$, are two real functionals of V, depending on $\lambda > 0$, we write

$$F_1(V;\lambda) \sim F_2(V;\lambda), \quad \lambda \downarrow 0,$$

if for each $\varepsilon \in (0,1)$ we have

$$F_2((1-\varepsilon)V;\lambda) + O_\varepsilon(1) \leq$$

$$F_1(V;\lambda) \leq$$

$$F_2((1+\varepsilon)V;\lambda) + O_\varepsilon(1).$$

Let $q \in \mathbb{Z}_+$. Then we have

$$\mathrm{Tr}\, \mathbf{1}_{(\Lambda_{q-1}, \Lambda_q - \lambda)}(H_L - V) \sim \mathrm{Tr}\, \mathbf{1}_{(\lambda, \infty)}(p_q V p_q), \tag{2}$$

with $\Lambda_{-1} = -\infty$, and

$$\mathrm{Tr}\, \mathbf{1}_{(\Lambda_q + \lambda, \Lambda_{q+1})}(H_L + V) \sim \mathrm{Tr}\, \mathbf{1}_{(\lambda, \infty)}(p_q V p_q), \tag{3}$$

as $\lambda \downarrow 0$.

Thus, *the Berezin-Toeplitz operator* $p_q V p_q$ is the effective Hamiltonian which governs the eigenvalue asymptotics of the operators

$$H_L \pm V$$

near the Landau level Λ_q, $q \in \mathbb{Z}_+$.

Steps of the proof of (2) - (3):

(i) *The Birman-Schwinger principle* implies

$$\operatorname{Tr} \mathbf{1}_{(\Lambda_q+\lambda, \Lambda_{q+1})}(H_L + V) =$$

$$\operatorname{Tr} \mathbf{1}_{(1,\infty)}(-V^{1/2}(H_L - \Lambda_q - \lambda)^{-1} V^{1/2}) + O(1),$$

$$\operatorname{Tr} \mathbf{1}_{(\Lambda_{q-1}, \Lambda_q - \lambda)}(H_L - V) =$$

$$\operatorname{Tr} \mathbf{1}_{(1,\infty)}(V^{1/2}(H_L - \Lambda_q + \lambda)^{-1} V^{1/2}) + O(1),$$

as $\lambda \downarrow 0$.

(ii) We have

$$\operatorname{Tr} \mathbf{1}_{(1,\infty)}(\pm V^{1/2}(H_L - \Lambda_q \pm \lambda)^{-1} V^{1/2}) \sim$$

$$\operatorname{Tr} \mathbf{1}_{(\lambda,\infty)}(V^{1/2} p_q V^{1/2}), \quad \lambda \downarrow 0,$$

since

$$\pm V^{1/2}(H_L - \Lambda_q \pm \lambda)^{-1} V^{1/2} =$$

$$\lambda^{-1} V^{1/2} p_q V^{1/2} \pm V^{1/2}(H_L - \Lambda_q \pm \lambda)^{-1}(I - p_q) V^{1/2},$$

and the second term admits a uniform limit as $\lambda \downarrow 0$.

(iii) Finally, for $\lambda > 0$,

$$\operatorname{Tr} \mathbf{1}_{(\lambda,\infty)}(V^{1/2} p_q V^{1/2}) = \operatorname{Tr} \mathbf{1}_{(\lambda,\infty)}(p_q V p_q),$$

since if $T \in S_\infty(\mathcal{H}_1, \mathcal{H}_2)$, then

$$\operatorname{Tr} \mathbf{1}_{(\lambda,\infty)}(T^* T) = \operatorname{Tr} \mathbf{1}_{(\lambda,\infty)}(T T^*), \quad \lambda > 0.$$

3.2. Basic properties of the Berezin-Toeplitz operators

Fix $q \in \mathbb{Z}_+$. Let $V \in L^\infty(\mathbb{R}^2)$. Then, evidently,

$$\|p_q V p_q\| \leq \|V\|_{L^\infty(\mathbb{R}^2)}.$$

Let now $V \in L^1(\mathbb{R}^2)$. Then the explicit expression for the value of the integral kernel of p_q on the diagonal easily implies

$$\|p_q V p_q\|_1 \leq \frac{b}{2\pi} \|V\|_{L^1(\mathbb{R}^2)}.$$

Interpolating, we find that if $r \in [1, \infty)$, and $V \in L^r(\mathbb{R}^2)$, then $p_q V p_q \in S_r(L^2(\mathbb{R}^2))$, and

$$\|p_q V p_q\|_r^r \leq \frac{b}{2\pi} \|V\|_{L^r(\mathbb{R}^2)}^r.$$

Moreover, if

$$V \in L^1_{\text{loc}}(\mathbb{R}^2), \qquad \lim_{|\mathbf{x}| \to \infty} V(\mathbf{x}) = 0,$$

then $p_q V p_q \in S_\infty(L^2(\mathbb{R}^2))$.

Proposition 1. *Let* $V \in L^1_{\text{loc}}(\mathbb{R}^2;\mathbb{R})$ *satisfy* $\lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = 0$. *Assume that* V *is radially symmetric, i.e. there exists* $v : [0,\infty) \to \mathbb{R}$ *such that* $V(\mathbf{x}) = v(|\mathbf{x}|)$, $x \in \mathbb{R}^2$. *Then the eigenvalues of the operator* $p_q V p_q$ *with domain* $p_q L^2(\mathbb{R}^2)$, *counted with the multiplicities, coincide with the set*

$$\frac{1}{k!}\int_0^\infty v((2t/B)^{1/2})\mathsf{L}_q(t)e^{-t}t^k dt, \quad k \in \mathbb{Z}_+.$$

Remark: If f is, say, a bounded function of exponential decay, then

$$(\mathcal{M}f)(z) := \int_0^\infty f(t)t^{z-1}dt, \; z \in \mathbb{C}, \; \mathrm{Re}\, z > 0,$$

is *the Mellin transform* of f.

3.3. Unitary equivalence of Berezin-Toeplitz operators and ΨDOs with Weyl symbols

Let $d \geq 1$, $s : \mathbb{R}^{2d} \to \mathbb{C}$ be an appropriate symbol. Then $\mathrm{Op}^w(s)$ denotes the pseudodifferential operator (ΨDO) acting in $L^2(\mathbb{R}^d)$, with Weyl symbol s, defined by

$$\left(\mathrm{Op}^w(s)u\right)(x) =$$

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} s\left(\frac{x+x'}{2}, \xi\right) e^{i(x-x')\cdot\xi} u(x') dx' d\xi.$$

Let us recall some of the properties of $\mathrm{Op}^w(s)$:
(i) Assume that

$$\|s\|_{\Gamma(\mathbb{R}^{2d})} :=$$

$$\sup_{\{\alpha,\beta\in\mathbb{Z}_+^d \mid |\alpha|,|\beta|\le[\frac{d}{2}]+1\}} \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\partial_x^\alpha \partial_\xi^\beta s(x,\xi)| < \infty.$$

Then $\mathrm{Op}^w(s) \in \mathcal{L}(L^2(\mathbb{R}^{2d}))$, and *the Calderón-Vaillancout-type estimates*

$$\|\mathrm{Op}^w(s)\| \le c_0 \|s\|_{\Gamma(\mathbb{R}^{2d})}$$

hold with a constant c_0 independent of s.
(ii) Let s_1, s_2 be two symbols which satisfy

$$s_2 = s_1 \circ \varkappa$$

with a *linear symplectomorphism* $\varkappa$. Then there exists a unitary operator $U = U(\varkappa)$ such that

$$\mathrm{Op}^w(s_2) = U^* \mathrm{Op}^w(s_1) U.$$

The operator $U(\varkappa)$, called *the metaplectic operator* corresponding to $\varkappa$, is defined uniquely up to a unimodular factor.

Introduce the harmonic oscillator

$$h := -\frac{d^2}{dx^2} + x^2,$$

self-adjoint in $L^2(\mathbb{R})$. We have

$$\sigma(h) = \bigcup_{q=0}^{\infty} \{2q+1\},$$

$$\dim \mathrm{Ker}\,(h - (2q+1)) = 1, \quad q \in \mathbb{Z}_+.$$

Let π_q be the orthogonal projection onto $\mathrm{Ker}(h-(2q+1))$, $q \in \mathbb{Z}_+$. We have

$$\pi_q = \langle \cdot, \varphi_q \rangle_{L^2(\mathbb{R})} \varphi_q$$

where

$$\varphi_q(x) := \frac{\mathrm{H}_q(x) e^{-x^2/2}}{(\sqrt{\pi} 2^q q!)^{1/2}}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

H_q being the Hermite polynomials. Let $2\pi\Psi_q$ be the Wigner function associated with φ_q, i.e. the Weyl symbol of π_q. For $(x,\xi) \in \mathbb{R}^2$ and $q \in \mathbb{Z}_+$, we have

$$\Psi_q(x,\xi) = \frac{(-1)^q}{\pi} \mathrm{L}_q(2(x^2+\xi^2)) e^{-(x^2+\xi^2)},$$

L_q being the Laguerre polynomials.

For $x \in \mathbb{R}^2$ set

$$V_b(x_1, x_2) = V(-b^{-1/2}x_2, -b^{-1/2}x_1).$$

Proposition 2. *There exists a unitary operator $\mathcal{U}_b : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ such that for any $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and $q \in \mathbb{Z}_+$, we have*

$$\mathcal{U}_b^* p_q V p_q \mathcal{U}_b = \pi_q \otimes \mathrm{Op}^w(V_b * \Psi_q).$$

Idea of the proof: Let s_1 be the symbol

$$(\xi_1 + bx_2/2)^2 + (\xi_2 - bx_1/2)^2\,,\ (x, \xi) \in T^*\mathbb{R}^2,$$

of the 2D Landau Hamiltonian, and s_2 is the symbol

$$b(\xi_1^2 + x_1^2)$$

of the operator $(bh) \otimes I$. Then there exists a linear symplectomorphism $\varkappa_b$ such that $s_2 = s_1 \circ \varkappa_b$. Then $\mathcal{U}_b$ is the metaplectic operator $U(\varkappa_b)$.

4. Spectral asymptotics for Berezin–Toeplitz operators

Based on the articles:

[RW] G.D.Raikov, S.Warzel, *Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials*, Rev. Math. Phys. **14** (2002), 1051-1072,

and

[R1990] G.D.Raikov, *Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips*, Commun. P.D.E. **15** (1990), 407-434.

4.1. Compactly supported symbols

Theorem 4. [RW] *Let* $0 \leq V \in L^\infty(\mathbb{R}^2)$*,* $\operatorname{supp} V$ *be compact, and* $V \geq C > 0$ *on an open non-empty subset of* $\mathbb{R}^2$*. Fix* $q \in \mathbb{Z}_+$*. Then*

$$\operatorname{Tr} \mathbf{1}_{(\lambda,\infty)}(p_q V p_q) = \varphi_\infty(\lambda)(1 + o(1)), \quad \lambda \downarrow 0, \tag{4}$$

where

$$\varphi_\infty(\lambda) := (\ln|\ln\lambda|)^{-1}|\ln\lambda|.$$

The proof of Theorem 4 is based on Proposition 1, and reduces finally to the asymptotics of integrals of the form

$$\frac{1}{k!}\int_0^R \mathsf{L}_q(t)e^{-t}t^k dt, \quad R \in (0,\infty), \quad q \in \mathbb{Z}_+,$$

as $k \to \infty$.

Remarks: (i) Relation (4) is not semiclassical in the sense that the function $\varphi_\infty(\lambda)$ is essentially different from

$$\mathcal{V}_q(\lambda) := \frac{1}{2\pi}\left|\left\{(x,\xi) \in \mathbb{R}^2 \,|(V_b * \Psi_q)(x,\xi) > \lambda\right\}\right|,$$

where $|\cdot|$ is the Lebesgue measure.

(ii) Let $\{\lambda_{n,q}\}_{n\in\mathbb{N}}$ be the non-increasing sequence of the positive eigenvalues of $p_q V p_q$, $q \in \mathbb{Z}_+$. Then (4) is equivalent to

$$\ln \lambda_{n,q} = -n \ln n\,(1 + o(1)), \quad n \to \infty.$$

In N. Filonov, A. Pushnitski, *Spectral asymptotics of Pauli operators and orthogonal polynomials in complex domains*, Comm. Math. Phys. **264** (2006), 759-772, it has been shown that under additional assumptions,

$$\ln \lambda_{n,q} =$$

$$-n \ln n + \left(1 + \frac{b\mathcal{C}(\mathrm{supp}\, V)^2}{2}\right) n(1 + o(1))$$

as $n \to \infty$, $\mathcal{C}(K)$ being *the logarithmic capacity* of a compact set $K \subset \mathbb{R}^2$.

Let us recall the definition of $\mathcal{C}(K)$. Let $\mathcal{M}(K)$ denote the set of probability measures on K. Then

$$\mathcal{C}(K) := e^{-\mathcal{I}(K)}$$

where

$$\mathcal{I}(K) := \inf_{\mu \in \mathcal{M}} \int_{K \times K} \ln |x - y|^{-1} d\mu(x) d\mu(y).$$

If K is simply connected, then $\mathcal{C}(K)$ is equal to the *conformal radius* of K.

4.2. Symbols of exponential decay

Theorem 5. [RW] *Let* $0 \leq V \in L^\infty(\mathbb{R}^2)$ *and*

$$\ln V(\mathbf{x}) = -\mu |\mathbf{x}|^{2\beta}(1 + o(1)), \quad |x| \to \infty, \tag{5}$$

with $\beta \in (0,\infty)$, $\mu \in (0,\infty)$. *Fix* $q \in \mathbb{Z}_+$. *Then*

$$\operatorname{Tr} \mathbf{1}_{(\lambda,\infty)}(p_q V p_q) = \varphi_{\beta,b}(\lambda)(1 + o(1)), \quad \lambda \downarrow 0, \tag{6}$$

where

$$\varphi_{\beta,b}(\lambda) := \begin{cases} \frac{b}{2\mu^{1/\beta}} |\ln \lambda|^{1/\beta} \text{ if } 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b)} |\ln \lambda| \text{ if } \beta = 1, \\ \frac{\beta}{\beta-1}(\ln|\ln \lambda|)^{-1} |\ln \lambda| \text{ if } 1 < \beta < \infty. \end{cases}$$

The proof of Theorem 6 is based on Proposition 1 and the following corollary from Proposition 2:

Corollary 3. *Let* $q \in \mathbb{Z}_+$, $V \in C_{\mathrm{b}}^{2q}(\mathbb{R}^2)$. *Then the operator* $p_q V p_q$ *is unitarily equivalent to* $p_0(\mathsf{L}_q(-\Delta/2b)V)p_0$, L_q *being the Laguerre polynomials.*

Thus, finally, the proof reduces to the asymptotics of integrals of the form

$$\frac{1}{k!}\int_0^\infty w_q((2t/b)^{1/2})e^{-t}t^k dt,$$

as $k \to \infty$. Here

$$w_q(r) :=$$

$$\mathsf{L}_q\left(-\frac{1}{2br}\frac{d}{dr}r\frac{d}{dr}\right)\left(e^{-\mu r^{2\beta}}\zeta(r)\right),\ r \in (0,\infty),\ q \in \mathbb{Z}_+,$$

and $\zeta \in C^\infty([0,\infty))$ is a cut-off function, vanishing near the origin, and identically equal to one outside a bounded interval.

Remarks: (i) If $0 < \beta < 1$, relation (6) is semiclassical, and if $\beta > 1$, it is not semiclassical. If $\beta = 1$, then the order of (6) is semiclassical, while the coefficient is not.

(ii) Relation (6) is equivalent to

$$\ln \lambda_{n,q} = \begin{cases} -\mu(2n/b)^{\beta}(1+o(1)),\ 0<\beta<1, \\ -(\ln(1+2\mu/b))n(1+o(1)),\ \beta=1, \\ -\frac{\beta-1}{\beta} n \ln n,\ \beta>1, \end{cases} \tag{7}$$

as $n \to \infty$. If we assume

$$\ln V(\mathbf{x}) = -\mu|\mathbf{x}|^{2\beta} + O(\ln|\mathbf{x}|), \quad |\mathbf{x}| \to \infty,$$

instead of (5), then we can improve (7) in the following way:

If $\beta \in (0,1)$, then there exist constants f_j, $j \in \mathbb{N}$, with $f_1 = \mu(2/b)^\beta$, such that

$$\ln \lambda_{n,q} =$$

$$-\sum_{1 \le j < \frac{1}{1-\beta}} f_j\, n^{(\beta-1)j+1} + O(\ln n), \quad n \to \infty.$$

If $\beta = 1$, then

$$\ln \lambda_{n,q} = -(\ln(1 + 2\mu/b)) + O(\ln n), \quad n \to \infty.$$

If $\beta \in (1,\infty)$, then there exist constants g_j, $j \in \mathbb{N}$, such that

$$\ln \lambda_{n,q} = -\frac{\beta-1}{\beta} n \ln n +$$

$$\left(\frac{\beta - 1 - \ln(\mu\beta(2/b)^\beta)}{\beta}\right) n -$$

$$\sum_{1 \le j < \frac{\beta}{\beta-1}} g_j\, n^{(\frac{1}{\beta}-1)j+1} + O(\ln n), \quad n \to \infty.$$

4.3. Symbols of power-like decay

Theorem 6. [R1990] *Let* $0 \le V \in C^1(\mathbb{R}^2)$, *and*

$$V(x) = v_0(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^{-\rho}(1 + o(1)),$$

$$|\nabla V(x)| = O(|\mathbf{x}|^{-\rho-1}),$$

as $|\mathbf{x}| \to \infty$, *with* $\rho > 0$, *and* $0 < v_0 \in C(\mathbb{S}^1)$. *Fix* $q \in \mathbb{Z}_+$. *Then*

$$\mathrm{Tr}\, \mathbf{1}_{(\lambda,\infty)}(p_q V p_q) =$$

$$\mathcal{V}_q(\lambda)(1 + o(1)) = \psi_{\rho,b}(\lambda)(1 + o(1)), \quad \lambda \downarrow 0, \tag{8}$$

where

$$\psi_{\rho,b}(\lambda) := \lambda^{-2/\rho} \frac{b}{4\pi} \int_0^{2\pi} v_0(\cos\theta, \sin\theta)^{2/\rho} d\theta.$$

The proof of Theorem 6 is based on Proposition 1.

Evidently, relation (8) is semiclassical. It is equivalent to

$$\lambda_{n,q} =$$

$$\left(\frac{b}{4\pi}\int_0^{2\pi} v_0(\cos\theta, \sin\theta)^{2/\rho} d\theta\right)^{\rho/2} n^{-\rho/2}(1+o(1)),$$

as $n \to \infty$.

4.4. Magnetic and metric perturbations

Up to now we discussed only electric perturbations of the 2D Landau Hamiltonian. Decaying perturbations of the magnetic field and of the metric are also of considerable interest, and the spectral asymptotics for such perturbations have been investigated by various authors.

Combined electric, magnetic and metric perturbation of power-like decay were considered V. Ivrii, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

Combined electric and magnetic perturbations of compact support were studied in G. Rozenblum, G. Tashchiyan, *On the spectral properties of the perturbed Landau Hamiltonian*, Comm. Partial Differential Equations **33** (2008), 1048–1081.

Results related to metric perturbations of power-like and exponential decay, or of compact support, can be found in T. Lungenstrass, G. Raikov, *Local spectral asymptotics for metric perturbations of the Landau Hamiltonian*, Analysis & PDE **8** (2015), 1237-1262.

Let us mention here also two articles which concern geometric perturbations of H_L:

A. Pushnitski, G. Rozenblum, *Eigenvalue clusters of the Landau Hamiltonian in the exterior of a compact domain*, Doc. Math. **12** (2007), 569–586.

The perturbed operator is the Landau Hamiltonian, but it acts in $L^2(\Omega)$ and is equipped with *Dirichlet* boundary conditions; here, $\Omega = \mathbb{R}^2 \setminus K$ with a suitable compact $K \subset \mathbb{R}^2$.

M. Persson, *Eigenvalue asymptotics of the even-dimensional exterior Landau-Neumann Hamiltonian*, Adv. Math. Phys. **2009**, Art. ID 873704, 15 pp.

The perturbed operator is the Landau Hamiltonian, but it acts in $L^2(\Omega)$ and is equipped with *Neumann* boundary conditions.

4.5. Extensions to 2D Pauli operators

Let $d = 2$. Assume that, say, b is a periodic magnetic field with non-zero mean value. Let

$$0 \leq V \in L^{\infty}(\mathbb{R}^2), \lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = 0.$$

Denote by $\wp$ the orthogonal projection onto $\mathrm{Ker}\, a$. Then, similarly to the Schrödinger case, we have

$$\mathrm{Tr}\, \mathbf{1}_{(-\infty,-\lambda)}(M(A,-V)) \sim \mathrm{Tr}\, \mathbf{1}_{(\lambda,\infty)}(\wp V \wp),$$

$$\mathrm{Tr}\, \mathbf{1}_{(\lambda,\mathcal{C})}(M(A,V)) \sim \mathrm{Tr}\, \mathbf{1}_{(\lambda,\infty)}(\wp V \wp),$$

as $\lambda \downarrow 0$.

Thus, we arrive at the problem of investigating the eigenvalue asymptotics of the compact Berezin-Toeplitz operator $\wp V \wp$.

Results on the asymptotics as $\lambda \downarrow 0$ of

$$\mathrm{Tr}\,\mathbf{1}_{(-\infty,-\lambda)}(M(A,-V)),\ \mathrm{Tr}\,\mathbf{1}_{(\lambda,\mathcal{C})}(M(A,V)),$$

for various b and V, are contained, for instance, in:

A. Iwatsuka, H. Tamura, *Asymptotic distribution of negative eigenvalues for two-dimensional Pauli operators with nonconstant magnetic fields*, Ann. Inst. Fourier **48** (1998), 479–515.

G.D.Raikov, *Spectral asymptotics for the perturbed 2D Pauli operator with oscillating magnetic fields. I. Non-zero mean value of the magnetic field*, Markov Processes and Related Fields **9** (2003), 775-794.

5. Singularities of the spectral shift function at the Landau levels

Based on the articles:

[BPR] V.Bruneau, A.Pushnitski, G.D.Raikov, *Spectral shift function in strong magnetic fields*, Algebra i Analiz **16** (2004), 207-238; translation in St. Petersburg Math. J., **16** (2005), 181-209,

[FR] C.Fernández, G.D.Raikov, *On the singularities of the magnetic spectral shift function at the Landau levels*, Ann. H. Poincaré **5** (2004), 381-403,

and

[R2006] G.D.Raikov, *Spectral shift function for magnetic Schrödinger operators*, Mathematical Physics of Quantum Mechanics, Lecture Notes in Physics, **690** (2006), 451-465.

5.1. The spectral shift function (SSF)

Let $d = 3$, $b > 0$, $\mathbf{B} = (0, 0, b)$, $\mathrm{curl}\, A = \mathbf{B}$. Let H_0 denote the 3D Landau Hamiltonian $H_L = H(A, 0)$. Assume

$$V \in C(\mathbb{R}^3; \mathbb{R}),\ |V(\mathbf{x})| \leq C \langle x_\perp \rangle^{-\rho_\perp} \langle x_{||} \rangle^{-\rho_{||}},$$

$$\mathbf{x} = (x_\perp, x_{||}),\ C \in [0, \infty), \rho_\perp > 2,\ \rho_{||} > 1. \quad (9)$$

Set $H := H_0 + V$. Then

$$(H - i)^{-1} - (H_0 - i)^{-1} \in S_1(L^2(\mathbb{R}^3)),$$

and there exists a unique

$$\xi = \xi(\cdot; H, H_0) \in L^1(\mathbb{R}; (1 + E^2)^{-1} dE)$$

such that *the Lifshits-Krein trace formula*

$$\mathrm{Tr}\,(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0) f'(E) dE$$

holds for each $f \in C_0^\infty(\mathbb{R})$, and

$$\xi(E; H, H_0) = 0$$

for each $E \in (-\infty, \inf \sigma(H))$.

The function $\xi(\cdot; H, H_0)$ is called *the spectral shift function* (SSF) for the operator pair (H, H_0).

For almost every $E > b = \inf \sigma_{\text{ac}}(H)$, the SSF $\xi(E; H, H_0)$ is proportional to *the scattering phase* for the operator pair (H, H_0), i.e. *Birman-Krein formula*

$$\det S(E; H, H_0) = e^{-2\pi i \xi(E; H, H_0)}$$

holds true, $S(E; H, H_0)$ being *the scattering matrix* for the operator pair (H, H_0).

Moreover, for almost every $E < b$ we have

$$\xi(E; H, H_0) = -\operatorname{Tr} \mathbf{1}_{(-\infty, E)}(H).$$

Properties of the SSF ([BPR]):

- $\xi(\cdot; H, H_0)$ is bounded on every compact subset of $\mathbb{R} \setminus (2b\mathbb{Z}_+ + b)$;

- $\xi(\cdot; H, H_0)$ is continuous on $\mathbb{R} \setminus ((2b\mathbb{Z}_+ + b) \cup \sigma_{\rm pp}(H))$ where $\sigma_{\rm pp}(H)$ is the set of the eigenvalues of H.

5.2. Asymptotics of the SSF near Λ_q

Our goal is to describe the asymptotic behaviour of the SSF $\xi(E; H, H_0)$ as $E \to \Lambda_q$, $q \in \mathbb{Z}_+$.

Let V satisfy (9). For $x_\perp \in \mathbb{R}^2$, $\lambda \geq 0$, set

$$W(x_\perp) := \int_{\mathbb{R}} |V(x_\perp, x_{||})| dx_{||},$$

$$\mathcal{W}_\lambda = \mathcal{W}_\lambda(x_\perp) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

where

$$w_{11} := \int_{\mathbb{R}} |V(x_\perp, x_{||})| \cos^2(\sqrt{\lambda} x_{||}) dx_{||},$$

$$w_{12} = w_{21} :=$$

$$\int_{\mathbb{R}} |V(x_\perp, x_{||})| \cos(\sqrt{\lambda} x_{||}) \sin(\sqrt{\lambda} x_{||}) dx_{||},$$

$$w_{22} := \int_{\mathbb{R}} |V(x_\perp, x_{||})| \sin^2(\sqrt{\lambda} x_{||}) dx_{||}.$$

We have

$$\operatorname{rank} p_q W p_q = \infty, \quad \operatorname{rank} p_q \mathcal{W}_\lambda p_q = \infty, \ \lambda \geq 0.$$

Theorem 7. [FR] *Let V satisfy*

$$|V(\mathbf{x})| \le C\langle \mathbf{x}\rangle^{-\rho},\ \mathbf{x} \in \mathbb{R}^3,\ \rho > 3,\ C \in [0,\infty). \tag{10}$$

Assume moreover that $V \ge 0$ or $V \le 0$. Fix $q \in \mathbb{Z}_+$. Then we have

$$\xi(\Lambda_q - \lambda; H, H_0) = O(1), \quad \lambda \downarrow 0,$$

if $V \ge 0$, and

$$\xi(\Lambda_q - \lambda; H, H_0) \sim -\mathrm{Tr}\, \mathbf{1}_{(2\sqrt{\lambda},\infty)}(p_q W p_q),\ \lambda \downarrow 0,$$

if $V \le 0$. Moreover,

$$\xi(\Lambda_q + \lambda; H, H_0) \sim \frac{1}{\pi}\mathrm{Tr}\ \arctan\left(\frac{p_q \mathcal{W}_\lambda p_q}{2\sqrt{\lambda}}\right),\ \lambda \downarrow 0,$$

if $V \ge 0$, and

$$\xi(\Lambda_q + \lambda; H, H_0) \sim -\frac{1}{\pi}\mathrm{Tr}\ \arctan\left(\frac{p_q \mathcal{W}_\lambda p_q}{2\sqrt{\lambda}}\right),\ \lambda \downarrow 0,$$

if $V \le 0$.

5.3. Proof of Theorem 7

A. Pushnitski's representation of the SSF (A. Pushnitski, *A representation for the spectral shift function in the case of perturbations of fixed sign*, St. Petersburg Math. J. **9** (1998), 1181–1194):

Assume that V satisfies (9). Then the norm limit

$$T(E) := \lim_{\delta \downarrow 0} |V|^{1/2}(H_0 - E - i\delta)^{-1}|V|^{1/2}$$

exists for every $E \in \mathbb{R} \setminus (2b\mathbb{Z}_+ + b)$. Moreover, $T(E)$ is compact, and

$$0 \leq \operatorname{Im} T(E) \in S_1.$$

Assume in addition that $\pm V \geq 0$. Then for $E \in \mathbb{R} \setminus (2b\mathbb{Z}_+ + b)$,

$$\xi(E; H, H_0) =$$

$$\pm\frac{1}{\pi}\int_{\mathbb{R}} \operatorname{Tr} \mathbf{1}_{(1,\infty)}(\mp(\operatorname{Re} T(E) + t \operatorname{Im} T(E))) \frac{dt}{1+t^2}.$$

Then we have

$$\pm\xi(E;H,H_0) \sim$$

$$\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\,\mathbf{1}_{(1,\infty)}(\mp(\mathrm{Re}\,T_q(E)+t\,\mathrm{Im}\,T_q(E)))\frac{dt}{1+t^2}$$

as $E \to \Lambda_q$ where, for $E \neq \Lambda_q$,

$$T_q(E) :=$$

$$\lim_{\delta\downarrow 0} |V|^{1/2}(p_q \otimes (H_{||} + \Lambda_q - E - i\delta)^{-1})|V|^{1/2}.$$

If $E = \Lambda_q - \lambda$ with $\lambda > 0$, then $T_q(E) = T_q(E)^* \geq 0$, and

$$\xi(E;H,H_0) = O(1), \quad \lambda \downarrow 0,$$

if $V \geq 0$, while

$$-\xi(E;H,H_0) \sim \mathrm{Tr}\,\mathbf{1}_{(1,\infty)}(T_q(\Lambda_q - \lambda)) \sim$$

$$\mathrm{Tr}\,\mathbf{1}_{(1,\infty)}\left(|V|^{1/2}\left(p_q \otimes \mathcal{O}_\lambda^-\right)|V|^{1/2}\right) =$$

$$\mathrm{Tr}\,\mathbf{1}_{(2\sqrt{\lambda},\infty)}(p_q W p_q), \quad \lambda \downarrow 0,$$

if $V \leq 0$, $\mathcal{O}_\lambda^-$ being the operator with constant integral kernel $1/(2\sqrt{\lambda})$.

If $E = \Lambda_q + \lambda$ with $\lambda \downarrow 0$, then

$$\pm\xi(E; H, H_0) \sim$$

$$\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\, \mathbf{1}_{(1,\infty)}(\mp t \mathrm{Im}\, T_q(E))\frac{dt}{1+t^2} =$$

$$\frac{1}{\pi}\mathrm{Tr}\ \arctan\left(\mathrm{Im}\, T_q(E)\right) =$$

$$\frac{1}{\pi}\mathrm{Tr}\ \arctan\left(\sqrt{|V|}\left(p_q \otimes \mathcal{O}_\lambda^+\right)\sqrt{|V|}\right) =$$

$$\frac{1}{\pi}\mathrm{Tr}\ \arctan\left(\frac{p_q \mathcal{W}_\lambda p_q}{2\sqrt{\lambda}}\right),$$

$\mathcal{O}_\lambda^+$ being the operator with integral kernel

$$\frac{\cos\sqrt{\lambda}(x_{\shortparallel} - x'_{\shortparallel})}{2\sqrt{\lambda}}, \quad x_{\shortparallel}, x'_{\shortparallel} \in \mathbb{R}.$$

5.4. A generalized Levinson formula

Corollary 4. [R2006] *Let V satisfy* (10), *and* $V \le 0$. *Fix $q \in \mathbb{Z}_+$. Then*

$$\lim_{\lambda \downarrow 0} \frac{\xi(\Lambda_q + \lambda; H, H_0)}{\xi(\Lambda_q - \lambda; H, H_0)} = \frac{1}{2\cos\frac{\pi}{\rho}}$$

if W admits a power-like decay with decay rate $\rho > 2$, or

$$\lim_{\lambda \downarrow 0} \frac{\xi(\Lambda_q + \lambda; H, H_0)}{\xi(\Lambda_q - \lambda; H, H_0)} = \frac{1}{2}$$

if W decays exponentially or has a compact support.

Remark: The classical Levinson formula relates the number of the negative eigenvalues of $-\Delta + V$, and $\lim_{E \downarrow 0} \xi(E; -\Delta + V, -\Delta)$.

5.5. Extensions to Pauli operators

Theorem 7 admits extensions to 3D Pauli operators with non constant magnetic fields $(0, 0, b)$, $b = b(x_\perp)$ being, say, an appropriate periodic magnetic field of non-zero mean value. In this case the role of the Landau levels is played by the origin.

The analogue of Theorem 7 for such Pauli operators can be found in G. Raikov, *Low energy asymptotics of the spectral shift function for Pauli operators with nonconstant magnetic fields*, Publ. Res. Inst. Math. Sci., **46** (2010), 645-670.

Related results for somewhat different magnetic fields of constant direction, and for negative energies (when the SSF is proportional to the eigenvalue counting function) are contained in A. Iwatsuka, H. Tamura, *Asymptotic distribution of eigenvalues for Pauli operators with nonconstant magnetic fields*, Duke Math. J. **93** (1998), 535–574.

6. Resonances near the Landau levels
Based on the articles:

[BBR2007] J.-F.Bony, V.Bruneau, G.D.Raikov, *Resonances and spectral shift function near the Landau levels*, Ann. Inst. Fourier, **57** (2007), 629-671,

[BBR2014] J.-F. Bony, V. Bruneau, G. Raikov, *Counting function of characteristic values and magnetic resonances*, Commun. P.D.E. **39** (2014), 274–305,

and

[BBR_Kyoto] J.-F. Bony, V. Bruneau, G. Raikov, *Resonances and spectral shift function singularities for magnetic quantum Hamiltonians*, In: Proceedings of the Conference Spectral and Scattering Theory and Related Topics, Kyoto, Japan, 2011, RIMS Kokyuroku Bessatsu **B45** (2014), 77-100.

We assume $d = 3$, and use the notations H_0, H as in the previous section.

6.1. Embedded eigenvalues

Theorem 8. [BBR_Kyoto] *Let*

$$|V|^{1/2}H_0^{-1/2} \in S_\infty(L^2(\mathbb{R}^3)).$$

Assume that V is axisymmetric, i.e. depends only on $|x_\perp|$ and $x_{||}$.
(i) *Let V satisfy*

$$-2b < V(\mathbf{x}) \leq -C\mathbf{1}_K(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3,$$

where $C > 0$, and $K \subset \mathbb{R}^3$ is an open non-empty set. Then each interval $(\Lambda_q, \Lambda_{q+1})$, $q \in \mathbb{Z}_+$, contains at least one (embedded) eigenvalue of H.
(ii) Let V satisfy

$$-2b < V(\mathbf{x}) \leq -C\mathbf{1}_{\tilde{K}}(x_\perp)\langle x_{||}\rangle^{-\rho_{||}},$$

$$\mathbf{x} = (x_\perp, x_{||}) \in \mathbb{R}^3,$$

where $C > 0$, $\rho_{||} \in (0,2)$, and $\tilde{K} \subset \mathbb{R}^2$ is an open non-empty set. Then each interval $(\Lambda_q, \Lambda_{q+1})$, $q \in \mathbb{Z}_+$, contains a sequence of eigenvalues of H which converges to Λ_{q+1}.

6.2. Meromorphic continuation of the resolvent of H and definition of resonances

For $z \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} \,|\, \text{Im}\, \zeta > 0\}$, we have

$$(H_0 - z)^{-1} = \sum_{q=0}^{\infty} p_q \otimes (H_{\shortparallel} + \Lambda_q - z)^{-1}\,.$$

Recall that the resolvent $(H_{\shortparallel} - z)^{-1}$ with $z \in \mathbb{C}_+$ admits the integral kernel

$$-\frac{e^{i\sqrt{z}|x_{\shortparallel} - x'_{\shortparallel}|}}{2i\sqrt{z}}, \quad x_{\shortparallel}, x'_{\shortparallel} \in \mathbb{R}, \quad \text{Im}\,\sqrt{z} > 0.$$

Let $\mathcal{M}$ be the infinite-sheeted Riemann surface of the family

$$\left\{\sqrt{z - \Lambda_q}\right\}_{q \in \mathbb{Z}_+},$$

and let

$$\mathcal{P} : \mathcal{M} \mapsto \mathbb{C} \setminus (2b\mathbb{Z}_+ + b)$$

be the corresponding covering.

For $\lambda_0 \in \mathbb{C}$ and $\varepsilon > 0$ put

$$D(\lambda_0, \varepsilon) := \{\lambda \in \mathbb{C} |\, |\lambda - \lambda_0| < \varepsilon\},$$

$$D(\lambda_0, \varepsilon)^* := \{\lambda \in \mathbb{C} |\, 0 < |\lambda - \lambda_0| < \varepsilon\}.$$

There exists an analytic bijection z_q:

$$D(0, \sqrt{2b})^* \ni k \mapsto z_q(k) \in D_q^* \subset \mathcal{M},$$

such that $\mathcal{P}(z_q(k)) = \Lambda_q + k^2$.

For $N > 0$ denote by $\mathcal{M}_N$ the part of $\mathcal{M}$ where $\operatorname{Im} \sqrt{z - \Lambda_q} > -N$ for all $q \in \mathbb{Z}_+$. Then, $\cup_{N>0} \mathcal{M}_N = \mathcal{M}$.

Proposition 3. [BBR2007]
(i) *For each* $N > 0$ *the operator-valued function*

$$(H_0 - z)^{-1} : e^{-N\langle x_{\shortparallel}\rangle} L^2(\mathbb{R}^3) \to e^{N\langle x_{\shortparallel}\rangle} L^2(\mathbb{R}^3)$$

has an analytic extension from $\mathbb{C}_+$ *to* $\mathcal{M}_N$.
(ii) *Suppose that* V *satisfies*

$$|V(\mathbf{x})| \leq C\langle x_\perp\rangle^{-\rho_\perp} \exp(-N|x_{\shortparallel}|),$$

$$\mathbf{x} = (x_\perp, x_{\shortparallel}) \in \mathbb{R}^3, \quad C \in [0, \infty), \tag{11}$$

for any $N > 0$ *and some* $\rho_\perp > 0$. *Then for each* $N > 0$ *the operator-valued function*

$$(H - z)^{-1} : e^{-N\langle x_{\shortparallel}\rangle} L^2(\mathbb{R}^3) \to e^{N\langle x_{\shortparallel}\rangle} L^2(\mathbb{R}^3),$$

has a meromorphic extension from $\mathbb{C}_+$ *to* $\mathcal{M}_N$ *whose poles and residue ranks do not depend on* N.

We define the resonances of H as the poles of the meromorphic extension of the resolvent $(H-z)^{-1}$, and denote their set by $\mathrm{Res}(H)$.

For $z_0 \in \mathrm{Res}(H)$ define its multiplicity by

$$\mathrm{mult}\,(z_0) := \mathrm{rank}\,\frac{1}{2i\pi}\int_\gamma (H-z)^{-1}dz,$$

where γ is an appropriate circle centered at z_0.

6.3. Resonance-free regions and regions with infinitely many resonances

Theorem 9. [BBR2007)] *Let* $0 < s_0 < \sqrt{2b}$ *and* $q \in \mathbb{Z}_+$. *Assume* V *satisfies* (11) *with* $\rho_\perp > 2$, *and is of definite sign* J. *Then for any* $\delta > 0$ *there exists* $g_0 > 0$ *such that:*
(i) $H_0 + gV$ *has no resonances in*

$$\{z = z_q(k) \,|\, 0 < |k| < s_0, \ -J\text{Im } k \leq \frac{1}{\delta}|\text{Re } k|\}$$

for any $0 \leq g \leq g_0$.
(ii) *If*

$$W(x_\perp) = \int_{\mathbb{R}} |V(x_\perp, x_{||})| dx_{||}$$

satisfies $\ln W(x_\perp) \leq -C\langle x_\perp \rangle^2$, *then for any* $0 < g \leq g_0$, *the operator* $H_0 + gV$ *has an infinite number of resonances in*

$$\{z = z_q(k) \,|\, 0 < |k| < s_0, \ -J\text{Im } k > \frac{1}{\delta}|\text{Re } k|\}.$$

The proof of Theorem (9) i based on the following

Proposition 4. *Suppose that* V *satisfies* (11) *with* $\rho_\perp > 2$. *Then* $z_0 \in \mathcal{M}$ *is a resonance of* H *if and only if* -1 *is an eigenvalue of*

$$\mathcal{T}_V(z_0) := \text{sign}\, V\, |V|^{1/2}(H_0 - z_0)^{-1}|V|^{1/2}.$$

Moreover,

$$\det{}_2\Big((H - z)(H_0 - z)^{-1}\Big) = \det{}_2\Big(I + \mathcal{T}_V(z)\Big),$$

has an analytic continuation from $\mathbb{C}_+$ *to* $\mathcal{M}$ *whose zeroes are the resonances of* H. *If* z_0 *is a resonance, then there exists a holomorphic function* $f(z)$, *for* z *close to* z_0, *such that* $f(z_0) \neq 0$ *and*

$$det_2\Big(I + \mathcal{T}_V(z)\Big) = (z - z_0)^{l(z_0)} f(z),$$

with $l(z_0) = \text{mult}\,(z_0)$.

6.4. Asymptotics of the resonance counting function

For $q \in \mathbb{Z}_+$ and $z \in D(0, \sqrt{2b})$ set

$$\mathcal{G}_q(z) := J|V|^{1/2}\Big(p_q \otimes \mathcal{O}_z\Big)|V|^{1/2} -$$

$$zJ|V|^{1/2} \sum_{j \neq q} \Big(p_j \otimes \Big(H_{||} + 2b(j-q) + z^2\Big)^{-1}\Big)|V|^{1/2},$$

$\mathcal{O}_z$ being the operator with integral kernel

$$\frac{1}{2} e^{z|x_{||} - x'_{||}|}, \qquad x_{||}, x'_{||} \in \mathbb{R}.$$

Let Π_q be the orthogonal projection onto $\operatorname{Ker} \mathcal{G}_q(0)$.

Theorem 10. [BBR2014] *Let V satisfy* (11) *with $\rho_\perp > 2$ and have a definite sign $J = \pm 1$. Let*

$$W(x_\perp) = \int_{\mathbb{R}} |V(x_\perp, x_{\shortparallel})| dx_{\shortparallel}, \quad x_\perp \in \mathbb{R}^2,$$

satisfy the assumptions of Theorem 4,5, or 6. Fix $q \in \mathbb{Z}_+$, and assume that $I - \mathcal{G}_q'(0)\Pi_q$ is invertible, Then for $0 < r_0 < \sqrt{2b}$ we have

$$\sum_{z_q(k) \in \mathrm{Res}(H) : r < |k| < r_0} \mathrm{mult}\,(z_q(k)) =$$

$$\mathrm{Tr}\, \mathbf{1}_{(2r,\infty)}(p_q W p_q)(1 + o(1))$$

as $r \downarrow 0$.

6.5. Proof of Theorem 10

(i) *Abstract results*

Let $\mathcal{D}$ be a domain of $\mathbb{C}$ containing 0, and let $\mathcal{H}$ be a separable Hilbert space. Consider the analytic function

$$G : \mathcal{D} \longrightarrow \mathcal{S}_\infty(\mathcal{H}).$$

Let $\Pi(G)$ be the orthogonal projection onto $\operatorname{Ker} G(0)$.

Assumptions:

$\mathcal{C}_1$: The operator $G(0)$ is self-adjoint;

$\mathcal{C}_2$: The operator $I - G'(0)\Pi(G)$ is invertible.

Let $\Omega \subset \mathcal{D} \setminus \{0\}$. Set

$$\mathcal{Z}_G(\Omega) = \Big\{\text{characteristic values of } G \text{ on } \Omega\Big\} :=$$

$$\Big\{z \in \Omega \,|\, I - \frac{G(z)}{z} \text{ is not invertible}\Big\}.$$

By $\mathcal{C}_1$, and $\mathcal{C}_2$ the set $\mathcal{Z}_G(\Omega)$ is discrete. For $z_0 \in \mathcal{Z}_G(\Omega)$ denote

$$\text{Mult}(z_0) :=$$

$$\frac{1}{2\pi i}\text{Tr}\int_\gamma \Big(I - \frac{G(z)}{z}\Big)'\Big(I - \frac{G(z)}{z}\Big)^{-1} dz$$

where γ is an appropriate circle centered at z_0. Set

$$\mathcal{N}_G(\Omega) := \sum_{z_0 \in \mathcal{Z}_G(\Omega)} \text{Mult}(z_0).$$

If $\partial\Omega$ is regular, and $\mathcal{Z}_G(\Omega) \cap \partial\Omega = \emptyset$, then

$$\mathcal{N}_G(\Omega) = \text{ind}_{\partial\Omega}\Big(I - \frac{G(z)}{z}\Big) :=$$

$$\frac{1}{2\pi i}\text{Tr}\int_{\partial\Omega} \Big(I - \frac{G(z)}{z}\Big)'\Big(I - \frac{G(z)}{z}\Big)^{-1} dz.$$

Proposition 5. *Assume* $\mathcal{C}_1$ *and* $\mathcal{C}_2$. *Suppose that the origin is an accumulation point of* $\mathcal{Z}_G(\mathcal{D} \setminus \{0\})$. *Then we have*

$$|\operatorname{Im} z_0| = o(|z_0|), \quad z_0 \in \mathcal{Z}_G(\mathcal{D} \setminus \{0\}),$$

as $z_0 \to 0$. *If, moreover,* $\pm G(0) \geq 0$, *then*

$$\pm \operatorname{Re} z_0 \geq 0$$

for $z_0 \in \mathcal{Z}_G(\mathcal{D} \setminus \{0\})$ *with* $|z_0|$ *small enough.*

For $0 < a < b < \infty$ and $\theta > 0$ set

$$C_\theta(a,b) := \{x + iy \in \mathbb{C} \,|\, a < x < b, \quad |y| < \theta x\}\,.$$

Proposition 6. *Assume $\mathcal{C}_1$ and $\mathcal{C}_2$. Suppose moreover that*

$$\mathrm{Tr}\, \mathbf{1}_{(r,\infty)}(G(0)) = \Phi(r)(1 + o(1)), \quad r \downarrow 0,$$

with $\Phi(r) = Cr^{-\gamma}$, or $\Phi(r) = C|\ln r|^\gamma$, or $\Phi(r) = C\frac{|\ln r|}{\ln|\ln r|}$, and some $\gamma, C > 0$. Then we have

$$\mathcal{N}_G(C_\theta(r,1)) = \Phi(r)(1 + o(1)), \quad r \downarrow 0,$$

for any $\theta > 0$.

(ii) *Sketch of the proof of Theorem 10*

- For $q \in \mathbb{N}$ and $k \in D(0, \sqrt{2b})^*$ we have

$$I + \mathcal{T}_V(z_q(k)) = I - \frac{\mathcal{G}_q(ik)}{ik}$$

i.e. $z_q(k) \in \text{Res}(H)$ if and only if ik is a characteristic value of $\mathcal{G}_q$. Moreover,

$$\text{mult}\,(z_q(k)) = \text{Mult}\,(ik).$$

- By Proposition 5 with $G = \mathcal{G}_q$, we have

$$\{z_q(k) \in \text{Res}(H) \,|\, r < |k| < r_0\} =$$

$$\{z_q(k) \in \text{Res}(H) \,|\, \pm ik \in C_\theta(r, r_0)\} + O(1),$$

as $r \downarrow 0$.

- Now the claim of Theorem 10 follows from Proposition 6 with $G = \mathcal{G}_q$ since

$$\text{Tr}\,\mathbf{1}_{(r,\infty)}(\mathcal{G}_q(0)) = \text{Tr}\,\mathbf{1}_{(2r,\infty)}(p_q W p_q).$$

7. High-energy behaviour of the spectral clusters for the 2D Landau Hamiltonian

Based on the articles:

[PRVB] A. Pushnitski, G. Raikov, C. Villegas-Blas, *Asymptotic density of eigenvalue clusters for the perturbed Landau Hamiltonian*, Commun. Math. Phys. **320** (2013), 425 - 453,

and

[LR] T. Lungenstrass, G. Raikov, *A trace formula for long-range perturbations of the Landau Hamiltonian*, Ann. H. Poincaré **15** (2014), 1523-1548.

7.1. Shrinking of the spectral clusters

Let $d = 2$, H_0 be the 2D Landau Hamiltonian in constant scalar magnetic field $b > 0$. Assume that V satisfies

$$V \in C(\mathbb{R}^2; \mathbb{R}), \quad |V(\mathbf{x})| \leq C\langle \mathbf{x} \rangle^{-\rho},$$
$$\mathbf{x} \in \mathbb{R}^2,\ C \in [0, \infty), \tag{12}$$

with some $\rho > 0$.

Theorem 11. (i) [PRVB] *Assume that* V *satisfies* (12) *with* $\rho > 1$. *Then*

$$\sigma(H) \subset \bigcup_{q \in \mathbb{Z}_+} \left(\Lambda_q - C_1 \lambda_q^{-\frac{1}{2}}, \Lambda_q + C_1 \lambda_q^{-\frac{1}{2}} \right)$$

with a constant $C_1 > 0$.

(ii) [LR] *Assume that* V *satisfies* (12) *with* $\rho \in (0, 1)$. *Then*

$$\sigma(H) \subset \bigcup_{q \in \mathbb{Z}_+} \left(\Lambda_q - C_2 \lambda_q^{-\frac{\rho}{2}}, \Lambda_q + C_2 \lambda_q^{-\frac{\rho}{2}} \right)$$

with a constant $C_2 > 0$.

7.2 Asymptotic density of the spectral clusters

(i) *Short–range V*

Assume that V satisfies (12) with $\rho > 1$. Define the Radon transform of V,

$$\tilde{V}(\omega, s) := \frac{1}{2\pi}\int_{\mathbb{R}} V(s\omega + t\omega^{\perp})dt,$$

where

$$\omega = (\omega_1, \omega_2) \in \mathbb{S}^1, \quad \omega^{\perp} = (-\omega_2, \omega_1), \quad s \in \mathbb{R}.$$

Note that assumption (12) with $\rho > 1$ entails

$$|\tilde{V}(\omega, s)| \leq C(1 + |s|)^{1-\rho}, \quad \omega \in \mathbb{S}^1, \quad s \in \mathbb{R}.$$

Theorem 12. [PRVB] *Let $V \in C(\mathbb{R}^2; \mathbb{R})$ satisfy* (12) *with $\rho > 1$. Then*

$$\lim_{q\to\infty} \Lambda_q^{-1/2} \operatorname{Tr} \varphi(\sqrt{\Lambda_q}(H - \Lambda_q)) =$$

$$\frac{1}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}} \varphi(\tilde{V}(\omega, s))\, ds\, d\omega \qquad (13)$$

for each $\varphi \in C_0^{\infty}(\mathbb{R} \setminus \{0\})$.

For $[\alpha,\beta]\subset\mathbb{R}\setminus\{0\}$ and $q\in\mathbb{Z}_+$ set

$$\mu_q^{\text{short}}([\alpha,\beta]) :=$$

$$\sum_{\Lambda_q+\alpha\Lambda_q^{-1/2}\le\lambda\le\Lambda_q+\beta\Lambda_q^{-1/2}} \dim\operatorname{Ker}(H-\lambda),$$

$$\mu_\infty^{\text{short}}([\alpha,\beta]) := \frac{1}{2\pi}\left|\tilde{V}^{-1}([b^{-1}\alpha, b^{-1}\beta])\right|.$$

Then (13) is equivalent to

$$\lim_{q\to\infty}\Lambda_q^{-1/2}\mu_q^{\text{short}}([\alpha,\beta]) = \mu_\infty^{\text{short}}([\alpha,\beta]),$$

for any α, β, such that $\alpha\beta>0$ and

$$\mu_\infty^{\text{short}}(\{\alpha\}) = \mu_\infty^{\text{short}}(\{\beta\}) = 0.$$

(ii) *Long–range* V

Let $\kappa \in \mathbb{R}$. We'll write:

- $u \in \mathcal{S}_1^{\kappa}(\mathbb{R}^2)$ if $u \in C^{\infty}(\mathbb{R}^2)$ satisfies
$$|D^{\alpha} u(x)| \leq C_{\alpha} \langle x \rangle^{\kappa - |\alpha|}, \quad \alpha \in \mathbb{Z}_+^2;$$

- $u \in \mathcal{H}_{\kappa}^{\sharp}(\mathbb{R}^2)$ if $u \in C^{\infty}(\mathbb{R}^2 \setminus \{0\})$ is homogeneous of order κ.

Assume that $\mathbb{V} \in \mathcal{H}_{-\rho}^{\sharp}(\mathbb{R}^2)$ with $\rho \in (0,1)$, and define its *mean–value* transform
$$\mathring{\mathbb{V}}(x) := \frac{1}{2\pi} \int_{\mathbb{S}^1} \mathbb{V}(x - \omega) d\omega, \quad x \in \mathbb{R}^2.$$
Since $\rho \in (0,1)$, we have $\mathring{\mathbb{V}} \in C(\mathbb{R}^2)$.

Moreover, $\sup_{x \in \mathbb{R}^2} |x|^{\rho} |\mathring{\mathbb{V}}(x)| < \infty$.

Theorem 13. *Let* $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$ *with* $\rho \in (0,1)$. *Assume that there exists* $\mathbb{V} \in \mathcal{H}^{\sharp}_{-\rho}(\mathbb{R}^2)$ *such that*

$$|V(\mathbf{x}) - \mathbb{V}(\mathbf{x})| \leq C|\mathbf{x}|^{-\rho-\varepsilon}, \ \mathbf{x} \in \mathbb{R}^2, \ |\mathbf{x}| > 1,$$

with some constant C *and* $\varepsilon > 0$. *Then*

$$\lim_{q\to\infty} \Lambda_q^{-1} \mathrm{Tr}\, \varphi(\Lambda_q^{\rho/2}(H - \Lambda_q)) =$$

$$\frac{1}{2\pi b}\int_{\mathbb{R}^2} \varphi(b^{\rho}\mathring{\mathbb{V}}(x))dx \qquad (14)$$

for each $\varphi \in C_0^{\infty}(\mathbb{R} \setminus \{0\})$.

For $[\alpha,\beta]\subset\mathbb{R}\setminus\{0\}$ and $q\in\mathbb{Z}_+$ set

$$\mu_q^{\mathrm{long}}([\alpha,\beta]) :=$$

$$\sum_{\Lambda_q+\alpha\Lambda_q^{-\rho/2}\le\lambda\le\Lambda_q+\beta\Lambda_q^{-\rho/2}} \dim\operatorname{Ker}(H-\lambda),$$

$$\mu_\infty^{\mathrm{long}}([\alpha,\beta]) := \frac{1}{2\pi b}\left|\mathring{\mathbb{V}}^{-1}([b^{-\rho}\alpha, b^{-\rho}\beta])\right|.$$

Then (14) is equivalent to

$$\lim_{q\to\infty}\Lambda_q^{-1}\mu_q^{\mathrm{long}}([\alpha,\beta]) = \mu_\infty^{\mathrm{long}}([\alpha,\beta]),$$

for any α, β, such that $\alpha\beta>0$ and

$$\mu_\infty^{\mathrm{long}}(\{\alpha\}) = \mu_\infty^{\mathrm{long}}(\{\beta\}) = 0.$$

7.3. Semiclassical interpretation

For $(\mathbf{x}, \xi) \in T^*\mathbb{R}^2$, consider the Hamiltonian function

$$\mathcal{H}(\boldsymbol{\xi}, \mathbf{x}) = \left(\xi_1 + \frac{1}{2} b x_2\right)^2 + \left(\xi_2 - \frac{1}{2} b x_1\right)^2 .$$

The projections of the orbits of the Hamiltonian flow of $\mathcal{H}$ onto the configuration space are circles of radius $\sqrt{E}/b$; here $E > 0$ is the energy corresponding to the orbit. The classical particles move around these circles with period $T_b = \pi/b$. The orbits can be parameterized by the energy $E > 0$ and the center $c \in \mathbb{R}^2$ of the circle. Let $\gamma(c, E, t)$, $t \in [0, T_b)$, be the path in the configuration space corresponding to such an orbit. Set

$$\text{Av}(V)(c, E) := \frac{1}{T_b} \int_0^{T_b} V(\gamma(c, E, t)) dt.$$

Then, under the hypotheses of Theorem 12 we have

$$\frac{1}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}}\varphi(\tilde{V}(\omega,s))ds\,d\omega =$$

$$\frac{b}{2\pi}\lim_{E\to\infty}\frac{1}{\sqrt{E}}\int_{\mathbb{R}^2}\varphi(\sqrt{E}\,\mathsf{Av}(V)(c,E))\,dc, \quad (15)$$

and the combination of (13) and (15) implies

$$\lim_{q\to\infty}\frac{1}{\sqrt{\Lambda_q}}\,\mathsf{Tr}\,\varphi(\sqrt{\Lambda}_q(H-\Lambda_q)) =$$

$$\frac{b}{2\pi}\lim_{E\to\infty}\frac{1}{\sqrt{E}}\int_{\mathbb{R}^2}\varphi(\sqrt{E}\,\mathsf{Av}(V)(c,E))\,dc. \quad (16)$$

Similarly, under the hypotheses of Theorem 13 we have

$$\frac{1}{2\pi b}\int_{\mathbb{R}^2}\varphi(b^\rho\mathring{\mathbb{V}}(x))dx =$$

$$\frac{b}{2\pi}\lim_{E\to\infty}\frac{1}{E}\int_{\mathbb{R}^2}\varphi(E^{\rho/2}\,\mathsf{Av}(V)(c,E))\,dc. \quad (17)$$

and the combination of (14) and (17) implies

$$\lim_{q\to\infty}\frac{1}{\Lambda_q}\,\mathrm{Tr}\;\varphi(\Lambda_q^{\rho/2}(H-\Lambda_q)) =$$

$$\frac{b}{2\pi}\lim_{E\to\infty}\frac{1}{E}\int_{\mathbb{R}^2}\varphi(E^{\rho/2}\,\mathsf{Av}(V)(c,E))\,dc. \quad (18)$$

Relations (16) and (18) could be interpreted in the spirit of the *averaging principle* for systems close to integrable ones (see e.g. Section 52 in V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics, **60** 1989).

According to this principle, a good approximation is obtained if one replaces the original perturbation by its average along the orbits of the free dynamics.

7.4. Proof of Theorem 11

We recall that Theorem 11 concerns the shrinking of the eigenvalue clusters of H to the Landau levels Λ_q.

First, we describe a suitable approximation of $\mathrm{Op}^w(V_b * \Psi_q)$. For $r > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$ define the distribution

$$\delta_r(\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \varphi(r\cos\theta, r\sin\theta)d\theta.$$

Proposition 7. [PRVB, LR] *Assume that* $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$ *with* $\rho > 0$. *Then*

$$\|\mathrm{Op}^w(V_b * \Psi_q) - \mathrm{Op}^w(V_b * \delta_{\sqrt{2q+1}})\|_2 = O(\Lambda_q^{-3/4}) \tag{19}$$

as $q \to \infty$.

Estimate (19) could be interpreted as equidistribution of the eigenfunctions of the harmonic oscillator h, i.e. as a weak convergence as $q \to \infty$ of the Wigner function $2\pi\Psi_q$ associated with the qth normalized eigenfunction of h, to the measure invariant with respect to the classical flow.

The key ingredient of the proof of Theprem 11 is:

Proposition 8. [PRVB, LR] *Assume that* $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$ *with* $\rho > 0$. *Then*

$$\| \operatorname{Op}^w(V_b * \delta_k)\| = \begin{cases} O(k^{-\rho}) & \text{if} \quad \rho \in (0,1), \\ O(k^{-1}\ln k) & \text{if} \quad \rho = 1, \\ O(k^{-1}) & \text{if} \quad \rho > 1, \end{cases}$$

as $k \to \infty$.

The proof is based on estimates of Calderón–Vaillancourt type for the norms of Weyl ΨDOs.

Applying Proposition 8 above, Proposition 2 on the unitary equivalence of $p_q V P_q$ and $\mathrm{Op}^w(V_b * \Psi_q)$, and Proposition 7 concerning the approximation of $\mathrm{Op}^w(V_b * \Psi_q)$ by $\mathrm{Op}^w(V_b * \delta_{\sqrt{2q+1}})$, we obtain the following:

Proposition 9. *Assume that V satisfies* (12) *with $\rho > 0$. Then*

$$\|p_q V p_q\| = \begin{cases} O(\Lambda_q^{-\rho/2}) & \text{if} \quad \rho \in (0,1), \\ O(\Lambda_q^{-1/2} \ln \Lambda_q) & \text{if} \quad \rho = 1, \\ O(\Lambda_q^{-1/2}) & \text{if} \quad \rho > 1, \end{cases} \tag{20}$$

as $q \to \infty$.

Theorem 11 now follows from Proposition 9, and the Birman–Schwinger principle.

7.5. Proof of Theorem 12

Let us recall that Theorem 12 concerns the asymptotic density of the eigenvalue clusters of H in the case of short–range V.

Although the reduction is highly non trivial, Theorem 12 follows from the Stone–Weierstrass theorem and the following:

Proposition 10. [PRVB] *Assume that* V *satisfies* (12) *with* $\rho > 1$. *Then*

$$\lim_{q\to\infty} \Lambda_q^{(\ell-1)/2}\,\mathrm{Tr}(p_q V P_q)^\ell =$$

$$\frac{b^\ell}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}} \tilde{V}(\omega,s)^\ell\,ds\,d\omega$$

for every integer $\ell > 1/(\rho-1)$.

The key ingredient of the proof of Proposition 10 is:

Proposition 11. [PRVB] *Let* $V \in C_0^\infty(\mathbb{R}^2)$*. Then for each* $\ell \in \mathbb{N}$ *we have*

$$\lim_{q \to \infty} \Lambda_q^{(\ell-1)/2} \, \mathrm{Tr} \; \mathrm{Op}^w(V_b * \delta_{\sqrt{2q+1}})^\ell =$$

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \tilde{V}(\omega, s)^\ell \, ds \, d\omega.$$

The proof is mainly based on the stationary phase method.

7.6. Proof of Theorem 13

Let us recall that Theorem 13 concerns the asymptotic density of the eigenvalue clusters of H in the case of long–range V.

The Dynkin-Helffer–Sjöstrand formula, suitable estimates in Schatten–von Neumann classes, and the Schur–Feshbach formula imply:

Proposition 12. *Under the hypotheses of Theorem 13 we have*

$$\mathrm{Tr}\,\varphi(\Lambda_q^{\rho/2}(H-\Lambda_q)) = \mathrm{Tr}\,\varphi(\Lambda_q^{\rho/2}p_qVp_q)+o(\Lambda_q)$$

as $q \to \infty$.

Theorem 13 now follows from Proposition 12 and the following:

Proposition 13. [LR] *Under the hypotheses of Theorem 13 we have*

$$\mathrm{Tr}\ \varphi(\Lambda_q^{\rho/2} p_q V p_q) =$$

$$\mathrm{Tr}\ \varphi(\Lambda_q^{\rho/2}\,\mathrm{Op}^w(\mathbb{V}_b * \delta_{\sqrt{2q+1}})) + o(\Lambda_q) \quad (21)$$

as $q \to \infty$, *and*

$$\lim_{q\to\infty} \Lambda_q^{-1}\,\mathrm{Tr}\ \varphi(\Lambda_q^{\rho/2}\,\mathrm{Op}^w(\mathbb{V}_b * \delta_{\sqrt{2q+1}})) =$$

$$\frac{1}{2\pi b}\int_{\mathbb{R}^2} \varphi(b^\rho \mathring{\mathbb{V}}(x))dx. \quad (22)$$

The proof of (21) employs suitable estimates in Schatten–von Neumann classes.

The proof of (22) is of semiclassical nature since $\Lambda_q^{\rho/2}\,\mathrm{Op}^w(\mathbb{V}_b * \delta_{\sqrt{2q+1}})$ is unitarily equivalent to the ΨDO with Weyl symbol

$$s_\hbar(x,\xi) := b^\rho \mathring{\mathbb{V}}_1(x, \hbar\xi), \quad (x,\xi) \in T^*\mathbb{R},$$

with

$$\hbar := \frac{1}{2q+1}.$$

However since this symbol is not smooth, a suitable approximation by smooth symbols is used at first, and then standard semiclassical techniques are applied.