Berezin-Toeplitz Operators in the Spectral and Scattering Theory of Magnetic Quantum Hamiltonians

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Abstract. The main goal of the course is to show the important role of the Berezin-Toeplitz operators in the spectral and scattering theory of magnetic quantum Hamiltonians.

First, I will consider the magnetic Schrödinger operator H, and will recall some of its basic properties such as the gauge invariance and the diamagnetic inequality. I will allocate special attention to the constant magnetic fields in space dimensions $d = 2, 3$ where the spectral properties of H differ drastically: as is well known, if $d = 2$, then the spectrum of H consists of an arithmetic progression of infinitely degenerated eigenvalues Λ_q, $q \in \mathbb{Z}_+$, called *Landau levels*, while if $d = 3$ the spectrum of H is absolutely continuous, and the Landau levels play the role of spectral thresholds. Further, in the case $d = 2$, I will consider the perturbed operator $H + V$ where V is a real-valued decaying potential, and will show that the effective Hamiltonian which governs the asymptotics of the eigenvalues of $H + V$ near a fixed Landau level Λ_q is the operator $P_q V P_q$ where P_q is the orthogonal projection onto $\mathrm{Ker}\,(H - \Lambda_q)$. $P_q V P_q$ can be identified as *a Berezin-Toeplitz operator* since the subspace $P_0 L^2(\mathbb{R}^2)$ is the Fock-Segal-Bargmann space of holomorphic functions, while the subspaces $P_q L^2(\mathbb{R}^2)$, $q \geq 1$, are obtained from $P_0 L^2(\mathbb{R}^2)$ by the action of the qth power of the magnetic creation operator.

Next, in the case $d = 3$, I will show that appropriate Berezin-Toeplitz operators play a crucial role in the analysis of the singularities of the Krein spectral shift function (SSF) for the operator pair $(H, H + V)$ with rapidly decaying V, and the asymptotic distribution of the resonances of $H + V$ near a fixed Landau level.

Finally, in the case $d = 2$, I will introduce the eigenvalue clusters around the Landau levels Λ_q, $q \in \mathbb{Z}_+$, for the operator $H + V$, and will show that these clusters shrink as $q \to \infty$. Moreover, I will present two trace formulae describing the asymptotic density of the eigenvalue clusters in the case of short-range and long-range perturbations V, respectively.

If time permits, extensions to Pauli and Dirac operators, as well as generalizations to magnetic and metric perturbations, will be briefly discussed.

Tentative Plan of the Course:

Lecture 1. Magnetic Schrödinger operator H in $L^2(\mathbb{R}^d)$, $d \geq 2$. Gauge invariance, diamagnetic inequality. Spectrum of H in constant magnetic fields for $d = 2, 3$. Magnetic Pauli operators in $L^2(\mathbb{R}^d; \mathbb{C}^2)$, $d = 2, 3$. Aharonov-Casher theorem and its generalizations. Magnetic Dirac operators.
Perturbations of H by decaying electric potentials V. The Berezin-Toeplitz operators as effective Hamiltonians governing the local spectral asymptotics for $H + V$ near the Landau levels in the case $d = 2$. Extensions to Pauli operators.
Unitary equivalence of the Berezin-Toeplitz operators and pseudo-differential operators of anti-Wick type.

Lecture 2. Eigenvalue asymptotics near the Landau levels for the operator $H + V$ in the case $d = 2$, and V of power-like decay, exponential decay, and compact support: transition from semi-classical to non semi-classical behavior. Extensions to Pauli operators. Generalizations for magnetic and metric perturbations.
Spectral shift function (SSF) for the operator pair $(H, H + V)$ in the case $d = 3$. A. Pushnitski's representation of the SSF. Basic properties of the SSF. Asymptotic behavior of the SSF near the Landau levels. A generalized Levinson formula. Extensions to Pauli and Dirac operators.
Embedded eigenvalues for the operator $H + V$ in the case $d = 3$. Meromorphic continuation of the resolvent of $H + V$ and definition of resonances. Resonance-free regions and regions with infinitely many resonances. Asymptotic distribution of the resonances near the Landau levels.

Lecture 3. The effect of the shrinking of the eigenvalue clusters for $H + V$ in the case $d = 2$ for short-range and long-range perturbations V. Radon transform of a short-range V and mean-value transform of homogeneous long range V. Asymptotic density of the eigenvalue clusters for $H + V$. Semi-classical interpretation in the spirit of the averaging principle.