GRECCHI, Vincenzo and MARTINEZ, André

Title

The spectral study of the cubic oscillator

Contents

Abstract

THE SPECTRAL STUDY OF THE CUBIC OSCILLATOR I: INTRODUCTION (A. Martinez)

This series of lectures concerns the cubic oscillator,

$$H(\beta) = -\frac{d^2}{dx^2} + x^2 + i\sqrt{\beta}x^3,$$

for β in the cut plane $\mathbb{C}_c := \mathbb{C}\backslash\mathbb{R}_-$.

In this first lecture, we will motivate the study by a brief review of various related arguments such as the Bessis-Zinn-Justin conjecture, the PT-symmetric operators, and the quartic oscillator (with results by Loeffel-Martin-Simon-Wightman). We will also state some few previous rigorous results on the cubic oscillator, together with a rough description of our recent result. Then, we will proceed with the description of basic techniques that will be used in the proof (Kato's perturbation theory, Sibuya's analysis, complex WKB method).

THE SPECTRAL STUDY OF THE CUBIC OSCILLATOR II: SUMMABILITY OF DIVERGENT SERIES (V. Grecchi)

We will review various classical ways of summing divergent series, such as the Borel method and the Stieljes-Padé summability in relation with the moment problem. In particular, we will recall Carleman's theorem of convergence for continuous fractions. Then, we will state our main result concerning the cubic oscillator, namely that, for β in the cut plane, the spectrum of $H(\beta)$ consists of the perturbative simple eigenvalues $\{E_n(\beta)\}_{n\geq 0}$ labeled by the constant number n of nodes of the corresponding eigenfunctions. In addition, for all $\beta \in \mathbb{C}_c$, $E_n(\beta)$ can be computed as the Stieltjes-Padé sum of its perturbation series at $\beta = 0$.

THE SPECTRAL STUDY OF THE CUBIC OSCILLATOR III: CONTROL OF THE NODESs (V. Grecchi)

We apply the perturbation theory to the cubic oscillator at $\beta = 0$ in order to obtain an asymptotic expansion of the eigenvalues of $H(\beta)$ for $|\beta|$ small. Next, we prove a general result of localization of the zeros of the eigenfunctions, namely the existence of a strip $\{0 \leq \text{Im } x \leq A(\beta)\}$ (with $A(\beta) > 0$) that contains no zero of the eigenfunctions of $H(\beta)$. This allows us to define the notion of nodes of an eigenfunction, and to state a result on their stability.

THE SPECTRAL STUDY OF THE CUBIC OSCILLATOR IV: FINAL ARGUMENTS (A. Martinez)

Combining the stability of the nodes with the complex WKB method, we prove the boundedness of the eigenvalues, first at $\beta \neq 0$, then at $\beta = 0$. This permits us to extend the eigenvalues as holomorphic functions of β in the whole cut plane. Then, checking their behaviors on the cut and at infinity, we prove that these functions are Stieltjes. Finally, applying Carleman's theorem, we deduce our final result.